

# Walks on the slit plane: other approaches

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## Abstract

Let  $\mathfrak{S}$  be a finite subset of  $\mathbb{Z}^2$ . A walk on the slit plane with steps in  $\mathfrak{S}$  is a sequence  $(0,0) = w_0, w_1, \dots, w_n$  of points of  $\mathbb{Z}^2$  such that  $w_{i+1} - w_i$  belongs to  $\mathfrak{S}$  for all  $i$ , and none of the points  $w_i$ ,  $i \geq 1$ , lie on the half-line  $\mathcal{H} = \{(k,0) : k \leq 0\}$ . In a recent paper, G. Schaeffer and the author computed the length generating function  $S(t)$  of walks on the slit plane for several sets  $\mathfrak{S}$ . All the generating functions thus obtained turned out to be algebraic: for instance, on the ordinary square lattice,

$$S(t) = \frac{(1 + \sqrt{1+4t})^{1/2}(1 + \sqrt{1-4t})^{1/2}}{2(1-4t)^{3/4}}.$$

The combinatorial reasons for this algebraicity remain obscure.

In this paper, we present two new approaches for solving slit plane models. One of them simplifies and extends the functional equation approach of the original paper. The other one is inspired by an argument of Lawler; it is more combinatorial, and explains the algebraicity of the product of three series related to the model. It can also be seen as an extension of the classical cycle lemma. Both methods work for any set of steps  $\mathfrak{S}$ .

We exhibit a large family of sets  $\mathfrak{S}$  for which the generating function of walks on the slit plane is algebraic, and another family for which it is neither algebraic, nor even D-finite. These examples give a hint at where the border between algebraicity and transcendence lies, and calls for a complete classification of the sets  $\mathfrak{S}$ .

## 1 Introduction

Let us consider square lattice walks that start from the origin  $(0,0)$ , but never return to the horizontal half-axis  $\mathcal{H} = \{(k,0) : k \leq 0\}$  once they have left their starting point: we call them *walks on the slit plane* (Figure 1). We denote by  $a_{i,j}(n)$  the number of such walks of length  $n$  that end at  $(i,j)$ . In [2], we proved that the associated generating function has a nice algebraic expression:

$$\begin{aligned} S(x,y;t) &:= \sum_{n \geq 0} t^n \sum_{(i,j) \in \mathbb{Z}^2} a_{i,j}(n) x^i y^j \\ &= \frac{(1 - 2t(1 + \bar{x}) + \sqrt{1-4t})^{1/2} (1 + 2t(1 - \bar{x}) + \sqrt{1+4t})^{1/2}}{2(1 - t(x + \bar{x} + y + \bar{y}))}, \end{aligned} \quad (1)$$

with  $\bar{x} = 1/x$  and  $\bar{y} = 1/y$ . We also studied the generating function for walks on the slit plane ending at a specific point  $(i,j)$ , proving, for instance, that  $a_{1,0}(2n+1) = C_{2n+1}$  and  $a_{0,1}(2n+1) = 4^n C_n$ , where  $C_n = \binom{2n}{n}/(n+1)$  is the  $n$ th Catalan number. These two formulae had been conjectured by Olivier Roques, following a question raised by Rick Kenyon on a mailing list. The enumeration of walks on the slit plane was studied previously in [3, 9], mostly in probabilistic terms, and is also related to [8]. However, all previously published results were asymptotic estimates, and to our knowledge, it had never been realized that the model could be exactly solved.

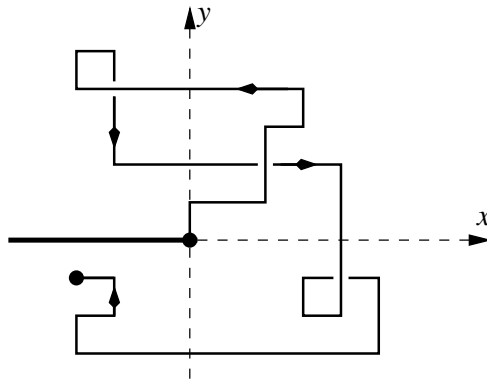


Figure 1: A walk on the slit plane.

Algebraic generating functions and Catalan numbers call for “classical” — if not bijective — combinatorial proofs (see [19, Ex. 6.19] for 66 other interpretations of these numbers). Indeed, the Florence group found a nice bijective proof of the  $C_{2n+1}$  result for walks on the slit plane ending at  $(1,0)$  [1]. However, their approach so far only works for this specific endpoint. As for the proof presented in [2], it is very atypical: from an obvious combinatorial argument, we derived a rather tricky functional equation, which we solved. This leaved us with the feeling that we had proved the identity (1) without understanding it completely. In particular, we have no combinatorial intuition as why this series is algebraic. Our functional equation approach has, nevertheless, also nice qualities: not only did it allow us to prove (1), but it worked just as well for walks taking their steps in a set  $\mathfrak{S}$  satisfying two simple conditions. For instance, we could solve the variation of the slit plane model illustrated in Figure 4. For all these sets  $\mathfrak{S}$ , we proved that the complete generating function  $S(x,y;t)$  (the counterpart of (1)) *and* the generating function for walks ending at a prescribed position  $(i,j)$  were always algebraic. This result has to be compared to the case of ordinary walks (say, on the square lattice): their generating function is rational, but, when restricted to walks ending at  $(i,j)$ , it becomes transcendental. In other words,

*forbidding a half-line adds some algebraicity to the model.*

This rather mysterious algebraicity is the main motivation for pursuing research on this topic: these results, we believe, need other explanations.

Let us confess right now that we shall not obtain here a completely satisfactory answer. We shall, however, explain why, for any set of steps  $\mathfrak{S}$ , the *product* of three generating functions counting certain walks on the slit plane is algebraic. We shall also clarify the limits of algebraicity, proving that it holds for a larger class of steps than in [2], but that it fails for other simple sets  $\mathfrak{S}$ , for which the generating function  $S(x,y;t)$  is not even D-finite<sup>1</sup>. These results rely on the fact that we are now able to solve slit plane problems *for any set*  $\mathfrak{S}$ , in terms of a certain factorization of the generating function for *bilateral walks* (walks ending on the  $x$ -axis, but otherwise unconstrained). This generating function is well-understood, and is always algebraic. Note that in these models, we only forbid the walks to step on the half-line  $\mathcal{H}$ , not to cross it. Examples are given in Figures 2, 4 and 7.

Let us now give a more detailed account of the contents of this paper. In Section 2, we introduce an elementary terminology on walks and words; we also recall some definitions and properties of algebraic and D-finite power series. The power series fans can certainly skip this part.

The paper really begins in Section 3. We present a “classical” combinatorial proof of (some) slit plane results. This proof is based on the so-called *cycle lemma*. The underlying idea is to recognize that the objects one wishes to count are in one-to-one correspondence with cyclic conjugacy classes of some other

<sup>1</sup>D-finite series, also called holonomic series, form a natural superset of algebraic series.

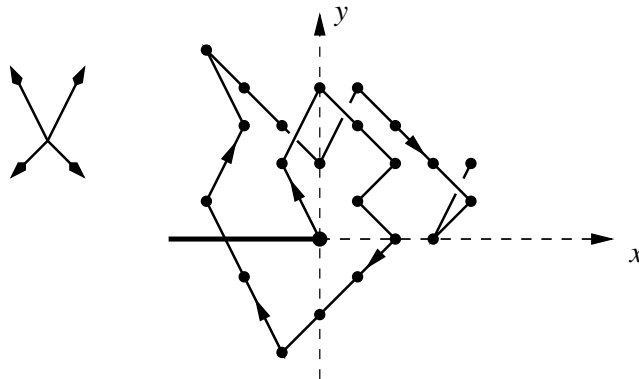


Figure 2: A walk on the slit plane with steps in  $\mathfrak{S} = \{(1,2), (-1,2), (1,-1), (-1,-1)\}$ .

objects, which are easier to enumerate. The most classical application of the cycle lemma is the enumeration of Dyck-like paths [4, 14]. It is interesting to note that the florentine proof of the  $C_{2n+1}$  formula for walks ending at  $(1,0)$ , although completely different from our approach, is also based on conjugacy of walks [1]. Our approach establishes a connection, valid for any set of steps  $\mathfrak{S}$ , between (some) walks on the slit plane and bilateral walks.

However, the cycle lemma suffers from a major drawback: it usually only works for walks ending at a specific position. In our case, we can only enumerate walks on the slit plane ending on the  $x$ -axis, as close as possible to the origin. This means at the point  $(1,0)$  for the models of Figures 1 and 2, and at the point  $(2,0)$  for the diagonal model of Figure 4.

The next section, Section 4, remedies this problem, even though its starting point lies very far from the cycle lemma. It is inspired by a “very nonintuitive” property that appears in Lawler’s book, *Intersections of random walks* [9], to which Greg Lawler drew my attention. This property, stated in probabilistic terms, asserts that

*for a random walk starting from  $(0,0)$ , with a killing rate<sup>2</sup>  $\lambda$ , the events  
“the walk avoids  $\{(k,0) : k < 0\}$ ” and “the walk avoids  $\{(k,0) : k \geq 0\}$ ”  
are independent.*

From this result, Lawler derived that the number of  $n$ -step walks on the slit plane, on the ordinary square lattice, grows like  $4^n n^{-1/4}$ . We translate this property into combinatorics, make a few elementary (but important) variations on it, and end up with an identity that relates the generating function  $S_0(x;t)$  for bilateral walks on the slit plane to the generating function  $B(x;t)$  for (ordinary) bilateral walks. Again, this identity is valid for any set of steps  $\mathfrak{S}$ . We prove that it contains and extends the cycle lemma result.

In Section 5, we prove that this identity completely determines the generating function for walks on the slit plane: given any set of steps, we are able to express  $S_0(x;t)$  and  $S(x,y;t)$  in terms of the *canonical factorization* of the generating function  $B(x;t)$  for bilateral walks. This kind of factorization also played a major role in [2], but for more mysterious reasons; the identity “à la Lawler” makes it completely natural. We argue that this expression of  $S(x,y;t)$  completely solves the problem: the series  $\log S(x,y;t)$  is always D-finite, and a differential equation defining it can be obtained in an algorithmic way.

In Section 6, we give an independent proof of the connection between  $S(x,y;t)$  and the canonical factorization of  $B(x;t)$ . We start from the very deep observation that an  $n$ -step walk on the slit plane is obtained by adding a step to an  $(n-1)$ -step walk on the slit plane. We translate this into a functional equation for  $S(x,y;t)$ , and solve this equation (for any set of steps  $\mathfrak{S}$ ) in a way that is so elementary that it looks fraudulent. This approach is also the one that was used in the original paper [2]: but the method we used for solving the functional equation was more complicated, and only worked for certain sets  $\mathfrak{S}$ .

<sup>2</sup>At each step, the walk is killed with probability  $\lambda$ .

The three previous sections provide tools for solving a specific model, given a set of steps  $\mathfrak{S}$ . They explain why  $\log S(x, y; t)$  is always D-finite. They do not explain why, on the square lattice,  $S(x, y; t)$  is algebraic. The next two sections of the paper are devoted to a partial exploration of the algebraic (and holonomic) properties of generating functions for walks on the slit plane. In Section 7, we generalize the result of [2] on algebraicity to all sets of steps  $\mathfrak{S}$  having *small height variations*: by this, we mean that adding a step to a walk never modifies its ordinate by more than one unit. This condition prevents a walk from crossing the forbidden half-line without hitting it. We solve explicitly three examples: the ordinary square lattice of Figure 1, the diagonal square lattice of Figure 4 (both already solved in [2]), and the triangular lattice of Figure 7. Would  $S(x, y; t)$  be algebraic for *all* sets of steps? No: we study in Section 8 some extensions of the diagonal square lattice for which  $S(x, y; t)$  is neither algebraic, nor even D-finite. An example is given in Figure 2.

We conclude the paper by mentioning possible directions of further research. This includes a complete classification of the sets  $\mathfrak{S}$  according to the algebraicity/transcendence of the corresponding generating function: an obvious conjecture is that algebraicity holds if and only if a walk cannot cross the forbidden half-line without hitting it. This also includes higher-dimensional models, to which the approach of this paper can be applied.

## 2 Preliminaries

### 2.1 Walks on the slit plane: definitions and notations

Let  $\mathfrak{S}$  be a finite subset of  $\mathbb{Z}^2$ . A walk with steps in  $\mathfrak{S}$  is a finite sequence  $w = (w_0, w_1, \dots, w_n)$  of vertices of  $\mathbb{Z}^2$  such that  $w_0 = (0, 0)$  and  $w_i - w_{i-1} \in \mathfrak{S}$  for  $1 \leq i \leq n$ . The number of steps,  $n$ , is the *length* of  $w$ . The *endpoint* of  $w$  is  $w_n$ , and its abscissa is denoted  $\delta(w)$ . The generating function for a set  $\mathcal{A}$  of walks is the formal series

$$A(x, y; t) = \sum_{n \geq 0} t^n \sum_{i, j \in \mathbb{Z}} a_{i, j}(n) x^i y^j,$$

where  $a_{i, j}(n)$  is the number of walks of  $\mathcal{A}$  that have length  $n$  and end at  $(i, j)$ . This series is a formal power series in  $t$  whose coefficients are polynomials in  $x, y, 1/x, 1/y$ . We shall often denote  $\bar{x} = 1/x$  and  $\bar{y} = 1/y$ .

We say that the walk  $w$  avoids the half-line  $\mathcal{H} = \{(k, 0), k \leq 0\}$  if none of the vertices  $w_1, \dots, w_n$  belong to  $\mathcal{H}$ . We call  $w$  a *walk on the slit plane* (Figure 2). We denote by  $\mathcal{S}$  the set of walks on the slit plane, and by  $S(x, y; t)$  the corresponding generating function. For  $j \in \mathbb{Z}$ , we denote by  $\mathcal{S}_j$  the set of walks on the slit plane ending at ordinate  $j$ , and by  $S_j(x; t)$  the corresponding (two-variable) generating function. Finally, for  $i, j \in \mathbb{Z}$ , we denote by  $\mathcal{S}_{i, j}$  the set of walks on the slit plane ending at the point  $(i, j)$ , and by  $S_{i, j}(t)$  the corresponding (one-variable) generating function.

We say that the walk  $w = (w_0, w_1, \dots, w_n)$  is a *loop* if  $w_0 = w_n = (0, 0)$  and none of the vertices  $w_i$  belong to the negative  $x$ -axis  $\{(k, 0), k < 0\}$ .

A walk with steps in  $\mathfrak{S}$  is called *bilateral* if it ends on the  $x$ -axis.

### 2.2 Walks and words

By definition, a walk is a sequence of steps of  $\mathfrak{S}$ . Hence, it will be convenient to consider walks as words on the alphabet  $\mathfrak{S}$ . The set of words  $\mathfrak{S}^*$  is equipped with the usual concatenation product. The empty word is denoted  $\epsilon$ . The number of occurrences of the letter  $a$  in the word  $w$  is denoted  $|w|_a$ . Any word  $w$  of  $\mathfrak{S}^*$  will be thought of as a walk starting from  $(0, 0)$ , and it will be convenient to make no distinction between the walk and the word, using expressions like “the endpoint of  $w$ ”, “the number of east steps in  $w$ ”, etc.

A *language* is a subset  $\mathcal{A}$  of  $\mathfrak{S}^*$ ; in other words, a set of walks. We denote by  $\mathbb{A}$  the *non-commutative generating function* of  $\mathcal{A}$ , that is, the formal power series

$$\mathbb{A} = \sum_{w \in \mathcal{A}} w.$$

The product  $\mathcal{A}_1 \mathcal{A}_2$  of two languages  $\mathcal{A}_1$  and  $\mathcal{A}_2$  is the set of words  $w$  that can be written as  $uv$ , where  $u \in \mathcal{A}_1$  and  $v \in \mathcal{A}_2$ . The product is *non-ambiguous* if any word in the product language can be factored in

a unique way as  $uv$ , with  $u \in \mathcal{A}_1$  and  $v \in \mathcal{A}_2$ . In this case, the non-commutative generating function for the product  $\mathcal{A}_1\mathcal{A}_2$  is  $\mathbb{A}_1\mathbb{A}_2$ .

## 2.3 Classes of formal power series

Given a ring  $\mathbb{L}$  and  $k$  indeterminates  $x_1, \dots, x_k$ , we denote by  $\mathbb{L}[x_1, \dots, x_k]$  the ring of polynomials in  $x_1, \dots, x_k$  with coefficients in  $\mathbb{L}$ , and by  $\mathbb{L}[[x_1, \dots, x_k]]$  the ring of *formal power series* in  $x_1, \dots, x_k$  with coefficients in  $\mathbb{L}$ , that is, formal sums

$$\sum_{n_1 \geq 0, \dots, n_k \geq 0} a_{n_1, \dots, n_k} x_1^{n_1} \cdots x_k^{n_k}, \quad (2)$$

where  $a_{n_1, \dots, n_k} \in \mathbb{L}$ . If  $\mathbb{L}$  is a field, we denote by  $\mathbb{L}(x_1, \dots, x_k)$  the field of rational functions in  $x_1, \dots, x_k$  with coefficients in  $\mathbb{L}$ . A *Laurent polynomial* in the  $x_i$  is a polynomial in the  $x_i$  and the  $\bar{x}_i = 1/x_i$ . A *Laurent series* in the  $x_i$  is a series of the form (2) in which the summation runs over  $n_1 \geq N_1, \dots, n_k \geq N_k$  for  $N_1, \dots, N_k \in \mathbb{Z}$ .

Let  $F$  be a series of  $\mathbb{L}[[x_1, \dots, x_k]]$ . Let  $i \leq k$  and write

$$F = \sum_{n \geq 0} F_n x_i^n$$

where the coefficients  $F_n$  are power series in the remaining variables  $x_\ell$ ,  $\ell \neq i$ . We denote by  $[x_i^n]F$  the coefficient of  $x_i^n$  in  $F$ , that is,  $[x_i^n]F = F_n$ . Let  $i, j \leq k$  and write

$$F = \sum_{m, n \geq 0} F_{m, n} x_i^m x_j^n$$

where the coefficients  $F_{m, n}$  are power series in the variables  $x_\ell$ ,  $\ell \neq i, j$ . The *diagonal* of  $F$  with respect to  $x_i$  and  $x_j$  is defined to be  $\sum_{n \geq 0} F_{n, n} x_i^n$ .

Assume  $\mathbb{L}$  is a field. A series  $F$  in  $\mathbb{L}[[x_1, \dots, x_k]]$  is *rational* if there exist non-trivial polynomials  $P$  and  $Q$  in  $\mathbb{L}[x_1, \dots, x_k]$  such that  $PF = Q$ . It is *algebraic* (over the field  $\mathbb{L}(x_1, \dots, x_k)$ ) if there exists a non-trivial polynomial  $P$  with coefficients in  $\mathbb{L}$  such that  $P(F, x_1, \dots, x_k) = 0$ . The sum and product of algebraic series is algebraic. The diagonals of a rational series are algebraic [19, p. 179].

The series  $F$  is *D-finite* (or *holonomic*) if the partial derivatives of  $F$  span a finite dimensional vector space on the field  $\mathbb{L}(x_1, \dots, x_k)$  (see [18] for the one-variable case, and [10, 11] otherwise). In other words, for  $1 \leq i \leq k$ , the series  $F$  satisfies a non-trivial partial differential equation of the form

$$\sum_{\ell=0}^{d_i} P_{\ell, i} \frac{\partial^\ell F}{\partial x_i^\ell} = 0,$$

where  $P_{\ell, i}$  is a polynomial in the  $x_j$ . Any algebraic series is holonomic. The sum and product of two holonomic series is still holonomic. The specializations of an holonomic series (obtained by giving values from  $\mathbb{L}$  to some of the variables) are holonomic, if well-defined. Moreover, if  $F$  is an *algebraic* series and  $G(t)$  is a holonomic series of one variable, then the substitution  $G(F)$  (if well-defined) is holonomic [11, Prop. 2.3]. Most importantly, if  $F$  is holonomic, then any diagonal of  $F$  is also D-finite [10].

## 3 Counting (some) walks on the slit plane by the cycle lemma

### 3.1 The ordinary square lattice

We focus here on the ordinary square lattice: the set of steps is  $\mathfrak{S} = \{(0, 1), (1, 0), (0, -1), (-1, 0)\} = \{o, n, e, s\}$ , the letter  $o$  (resp.  $n, e, s$ ) standing for a west (resp. north, east, south) step<sup>3</sup>. Recall that a

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<sup>3</sup>The french word for *west* being *ouest*.

*bilateral walk* is a walk that ends on the  $x$ -axis. A (bicolored) *Motzkin walk* is a bilateral walk that never visits a point with a negative ordinate. We shall enumerate these walks not only by their length and endpoint (variables  $t$  and  $x$ ), but also by the number of their vertical steps, using an additional indeterminate  $v$ . The associated generating functions can be determined by a standard argument.

**Lemma 1** *The generating function for bicolored Motzkin walks is*

$$M(x; t, v) = \frac{1 - t(x + \bar{x}) - \sqrt{[1 - t(x + \bar{x} + 2v)][1 - t(x + \bar{x} - 2v)]}}{2t^2v^2}.$$

*The generating function for bilateral walks is*

$$B(x; t, v) = \frac{1}{\sqrt{[1 - t(x + \bar{x} + 2v)][1 - t(x + \bar{x} - 2v)]}}.$$

**Proof.** The simplest approach consists in factoring (non-empty) Motzkin walks at the first place where they return to the horizontal axis. This gives, for their non-commutative generating function,

$$\mathbb{M} = \epsilon + o \mathbb{M} + e \mathbb{M} + n \mathbb{M} s \mathbb{M},$$

so that

$$M(x; t, v) = 1 + t(x + \bar{x})M(x; t, v) + t^2v^2M(x; t, v)^2.$$

The expression of  $M(x; t, v)$  follows. The same principle, applied to bilateral walks, gives

$$\mathbb{B} = \epsilon + o \mathbb{B} + e \mathbb{B} + n \mathbb{M} s \mathbb{B} + s \Phi(\mathbb{M}) n \mathbb{B},$$

where  $\Phi$  is the morphism that flips the walks around the horizontal axis:  $\Phi(n) = s$ ,  $\Phi(s) = n$ ,  $\Phi(e) = e$  and  $\Phi(o) = o$ . Hence

$$B(x; t, v) = 1 + t(x + \bar{x})B(x; t, v) + 2t^2v^2B(x; t, v)M(x; t, v).$$

The expression of  $B(x; t, v)$  follows. ■

Recall that the  $n$ th Catalan number  $C_n$  is defined by:

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

**Proposition 2** *The number of walks of length  $2n+1$  on the slit plane ending at  $(1,0)$  is the Catalan number  $C_{2n+1}$ . More precisely, the number of such walks having exactly  $2m$  vertical steps is*

$$4^m \binom{2n}{2m} C_{n-m}.$$

*This is also the number of bicolored Motzkin walks of length  $2n$  with  $2m$  horizontal steps.*

**Proof.** Let us say that a non-empty bilateral walk  $w$  is *primitive* if the only vertices of  $w$  that lie on the  $x$ -axis are its endpoints. We denote by  $\mathcal{P}$  the set of primitive words, and by  $P(x; t, v)$  their generating function.

Let  $w \in \mathcal{S}_{1,0}$ , and assume that exactly  $(k+1)$  of its vertices lie on the  $x$ -axis. Then  $w$  can be written in a unique way as the concatenation  $w_1w_2 \cdots w_k$  of  $k$  primitive words. For  $i \in [k]$ , let  $w^{(i)}$  be obtained by permuting cyclically the factors  $w_\ell$ , starting from  $w_i$  (see Figure 3). That is,  $w^{(i)} = w_iw_{i+1} \cdots w_kw_1 \cdots w_{i-1}$ . Note that  $\delta(w^{(i)}) = \delta(w) = 1$ .

Conversely, let  $v = v_1 \cdots v_k$  be a word of  $\mathcal{P}^k$  such that  $\delta(v) = 1$ . Let  $v_j$  be the primitive factor following the last vertex of  $v$  that lies on the  $x$ -axis and has a minimal abscissa. Clearly,  $v^{(j)} := v_jv_{j+1} \cdots v_kv_1 \cdots v_{j-1}$  belongs to  $\mathcal{S}_{1,0}$ . We claim that no other word  $v^{(\ell)}$  belongs to this set. Indeed, if  $\ell < j$ , then the prefix

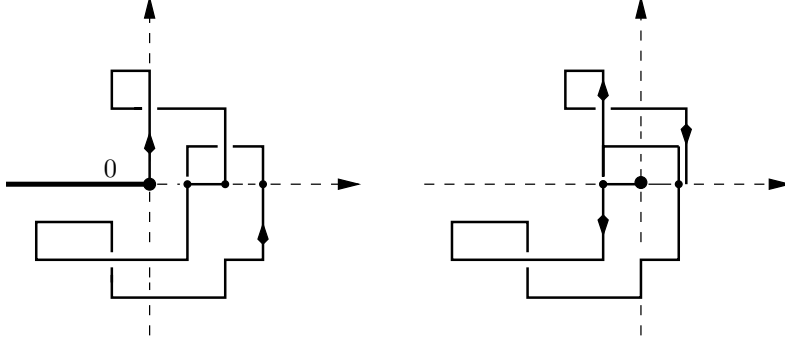


Figure 3: A word  $w = w_1 \cdots w_4$  of  $\mathcal{S}_{1,0}$  and one of its conjugates  $v = w_2 w_3 w_4 w_1 = v_1 v_2 v_3 v_4$ .

$v_\ell \cdots v_{j-1}$  of  $v^{(\ell)}$  ends at abscissa  $\delta(v_1 \cdots v_{j-1}) - \delta(v_1 \cdots v_{\ell-1})$ . By definition of  $j$ , this abscissa is non-positive, so that this prefix of  $v^{(\ell)}$  ends on  $\mathcal{H}$ . Similarly, if  $\ell > j$ , then  $v_\ell \cdots v_k v_1 \cdots v_{j-1}$  is a prefix of  $v^{(\ell)}$  that ends at abscissa  $1 - \delta(v_1 \cdots v_{\ell-1}) + \delta(v_1 \cdots v_{j-1}) < 1$ , that is, on the forbidden half-line  $\mathcal{H}$ .

This proves that the map  $(w, i) \mapsto w^{(i)}$  establishes a one-to-one correspondence between:

- pairs  $(w, i)$ , where  $w$  is a walk of  $\mathcal{S}_{1,0}$  formed of  $k$  primitive factors and  $i \in [1, k]$ , and
- bilateral walks  $v$  ending at  $(1, 0)$  and formed of  $k$  primitive factors.

This bijection preserves the number of vertical/horizontal steps. Consequently, the generating function for walks of  $\mathcal{S}_{1,0}$  formed of  $k$  primitive factors is

$$\frac{1}{k} [x] P(x; t, v)^k,$$

and summing over  $k \geq 1$  gives

$$S_{1,0}(t, v) = [x] \sum_{k \geq 1} \frac{P(x; t, v)^k}{k} = [x] \log \frac{1}{1 - P(x; t, v)} = [x] \log B(x; t, v),$$

because a bilateral walk is simply a sequence of primitive walks. The explicit value of  $B(x; t, v)$ , given in Lemma 1, yields

$$\begin{aligned} S_{1,0}(t, v) &= \frac{1}{2} [x] \left[ \log \frac{1}{1 - t(x + \bar{x} + 2v)} + \log \frac{1}{1 - t(x + \bar{x} - 2v)} \right] \\ &= \frac{1}{2} [x] \sum_{n \geq 1} \frac{t^n}{n} [(x + \bar{x} + 2v)^n + (x + \bar{x} - 2v)^n] \\ &= [x] \sum_{n \geq 1} \frac{t^n}{n} \sum_{m \geq 0} (2v)^{2m} \binom{n}{2m} (x + \bar{x})^{n-2m} \\ &= \sum_{n \geq 0} \frac{t^{2n+1}}{2n+1} \sum_{m=0}^n (2v)^{2m} \binom{2n+1}{2m} \binom{2n-2m+1}{n-m}. \end{aligned}$$

This gives the second result of Proposition 2. Now when  $v = 1$ ,

$$B(x; t, 1) = \frac{1}{\sqrt{(1 - t\bar{x}(1+x)^2)(1 - t\bar{x}(1-x)^2)}}$$

and a similar calculation yields the first result of the proposition. ■

**Remark.** It is easy to see that  $4^m \binom{2n}{2m} C_{n-m}$  is the number of bicolored Motzkin words of length  $2n$  with  $2m$  horizontal steps: such words are obtained by shuffling a word of length  $2m$  on the alphabet  $\{e, o\}$  with a Dyck word of length  $2n - 2m$  on the alphabet  $\{n, s\}$ . And Dyck words are well-known to be counted by Catalan numbers. A nice bijection between walks of  $\mathcal{S}_{1,0}$  and bicolored Motzkin words proving Proposition 2 is presented in [1].

### 3.2 Walks with steps in an arbitrary set $\mathfrak{S}$

We first focus on bilateral walks: whatever the set  $\mathfrak{S}$ , their generating function is algebraic.

**Lemma 3** *Let  $K(x, y; t)$  be the following polynomial:*

$$K(x, y; t) = 1 - t \sum_{(i,j) \in \mathfrak{S}} x^i y^j. \quad (3)$$

*The generating function for bilateral walks is algebraic, given by*

$$B(x; t) = [y^0] \frac{1}{K(x, y; t)}.$$

**Proof.** A walk being a sequence of steps, the generating function  $W(x, y; t)$  for all walks with steps in  $\mathfrak{S}$  is the rational function  $W(x, y; t) = 1/K(x, y; t)$ . The expression of  $B$  is a direct consequence of the definition of bilateral walks.

Let us now discuss the algebraic nature of  $B$ . The idea is that taking a constant term is equivalent to taking a diagonal<sup>4</sup>. We introduce two new variables,  $\bar{x}$  and  $\bar{y}$ , independent, for once, of  $x$  and  $y$ . We refine the weight of the walks by giving to a step  $(i, j)$  the weight  $t x^i y^j$  if  $i \geq 0$  and  $j \geq 0$ , the weight  $t \bar{x}^{-i} y^j$  if  $i \leq 0$  and  $j \geq 0$ , etc. Let  $\overline{W}(x, \bar{x}, y, \bar{y}; t)$  be the generating function for all walks, in which each walk is weighted by the product of the weights of its steps. This series is still rational, and  $B(x; t)$  is derived from  $\overline{W}$  first by taking the diagonal with respect to  $y$  and  $\bar{y}$  (which yields an algebraic series), then by setting  $y = 1$  and  $\bar{x} = 1/x$  (which preserves algebraicity). ■

We now state the generalization of Proposition 2.

**Proposition 4 (The cycle lemma result)** *Let  $p$  be the smallest positive integer such that there exists a walk on the slit plane ending at the point  $(p, 0)$ . Then the generating function for such walks is*

$$S_{p,0}(t) = [x^p] \log B(x; t)$$

*where  $B(x; t)$  counts bilateral walks. This series is  $D$ -finite.*

**Proof.** The proof mimics that of Proposition 2, with one subtlety. Again, we say that a non-empty bilateral walk  $w$  is primitive if the only vertices of  $w$  that lie on the  $x$ -axis are its endpoints.

Let  $v = v_1 \cdots v_k$  be a bilateral word formed of  $k$  primitive factors, and such that  $\delta(v) = m > 0$ . Let  $v_j$  be the primitive factor following the last vertex of  $v$  that lies on the  $x$ -axis and has a minimal abscissa. Clearly,  $v^{(j)} = v_j v_{j+1} \cdots v_k v_1 \cdots v_{j-1}$  belongs to  $\mathcal{S}_{m,0}$ . In particular,  $m \geq p$ . This shows that *a bilateral walk ending at a positive abscissa actually ends to the right of  $(p, 0)$ .*

We can now restrict the study to the case where the endpoint of  $v$  is exactly  $(p, 0)$ . We claim that no other word  $v^{(\ell)}$  belongs to  $\mathcal{S}_{p,0}$ . Indeed, if  $\ell < j$ , then the prefix  $v_\ell \cdots v_{j-1}$  of  $v^{(\ell)}$  ends at abscissa  $\delta(v_1 \cdots v_{j-1}) - \delta(v_1 \cdots v_{\ell-1})$ . By definition of  $j$ , this abscissa is non-positive, so that this prefix of  $v^{(\ell)}$  ends on  $\mathcal{H}$ . Similarly, if  $\ell > j$ , then  $v_\ell \cdots v_k v_1 \cdots v_{j-1}$  is a prefix of  $v^{(\ell)}$  that ends at abscissa  $p - \delta(v_1 \cdots v_{\ell-1}) + \delta(v_1 \cdots v_{j-1}) < p$ . But we have just proved that this forces this prefix (a bilateral walk) to end at a non-positive abscissa, hence on  $\mathcal{H}$ .

From this point, we conclude exactly as in the proof of Proposition 2 to obtain the expression of  $S_{p,0}$ .

---

<sup>4</sup>An argument that goes against nature: several proofs that the diagonal of a rational series is algebraic start by transforming the diagonal problem into a constant term one...



Let us now discuss the holonomic nature of this series. As in the proof of Lemma 3, we refine the weight of the steps by introducing two new variables  $\bar{x}$  and  $\bar{y}$ , independent of  $x$  and  $y$ . The generating function  $\overline{B}(x, \bar{x}; t)$  that counts bilateral walks is the diagonal of a rational series, and thus is an algebraic series in  $x$ ,  $\bar{x}$  and  $t$ . Hence its logarithm is D-finite. Finally,  $S_{p,0}(t)$  is obtained by first taking the diagonal of the series  $\bar{x}^p \log B(x, \bar{x}; t)$  with respect to  $x$  and  $\bar{x}$ , which yields a D-finite series, and then by setting  $x = 1$ , which preserves holonomy. ■

The fact that we are dealing with two-dimensional walks should not hide that our argument is essentially one-dimensional. Let us state explicitly the result for walks on  $\mathbb{Z}$  that underlies Proposition 4.

**Lemma 5** *Let  $\mathfrak{P}$  be a set of steps in  $\mathbb{Z}$ , and let  $\lambda = (\lambda_i)_{i \in \mathfrak{P}}$  be a sequence of variables describing the weights of these steps. We define the weight of a path  $w$  with steps in  $\mathfrak{P}$  to be the product of the weights of its steps. Let  $p$  be the smallest positive integer such that there exists a walk ending at  $p$ . Then the generating function for  $n$ -step walks that start from 0, end at  $p$  and always have a positive level once they have left their starting point is*

$$\frac{1}{n} [x^p] \left( \sum_{i \in \mathfrak{P}} \lambda_i x^i \right)^n.$$

Hence the generating function for walks ending at  $p$  is

$$S_p = [x^p] \log \frac{1}{1 - \sum_{i \in \mathfrak{P}} \lambda_i x^i}.$$

### 3.3 The diagonal square lattice

We now apply the general result of Proposition 4 to the diagonal version of the square lattice, illustrated in Figure 4. The set of steps is  $\mathfrak{S} = \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$ . By analogy with what we did on the square lattice, we put an additional weight  $v$  on the “vertical” steps  $(1, 1)$  and  $(-1, -1)$ . This, we shall see, gives a new occurrence of the Narayana distribution [12, 19, p. 273].

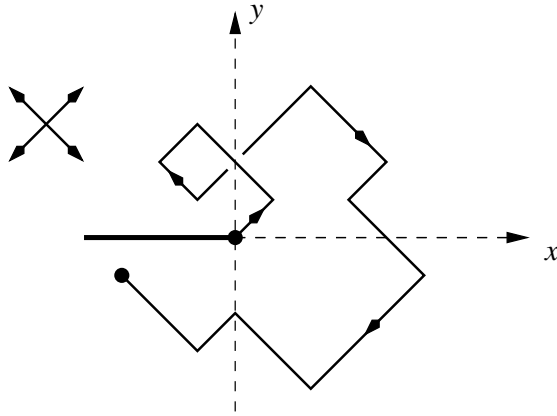


Figure 4: A walk on the slit plane with diagonal steps.

**Lemma 6** *The generating function for bilateral walks on the diagonal square lattice is*

$$B(x; t, v) = \frac{1}{\sqrt{1 - 4t^2(v + x^2)(v + \bar{x}^2)}}.$$

**Proof.** The principle is the same as in Lemma 1. The generating function  $M(x; t, v)$  for bilateral walks that stay on or above the  $x$ -axis satisfies

$$M(x; t, v) = 1 + t(xv + \bar{x})M(x; t, v)t(x + \bar{x}v)M(x; t, v) = 1 + t^2(v + x^2)(v + \bar{x}^2)M(x; t, v)^2.$$

The generating function  $B$  for bilateral walks is related to  $M$  by

$$B(x; t, v) = 1 + 2t^2(v + x^2)(v + \bar{x}^2)M(x; t, v)B(x; t, v).$$

(The factor 2 comes from the fact that primitive bilateral words starting with a down step are obtained by reading from right to left primitive bilateral words starting with an up step.) The results follows. ■

**Proposition 7** *The number of walks of length  $2n$  with diagonal steps going from  $(0, 0)$  to  $(2, 0)$  on the slit plane is*

$$\frac{1}{2} 4^n C_n$$

where  $C_n$  denotes the  $n$ th Catalan number. More precisely, the number of such walks having exactly  $2m - 1$  steps in the set  $\{(-1, -1), (1, 1)\}$  is

$$\frac{1}{2} \frac{4^n}{n} \binom{n}{m} \binom{n}{m-1}.$$

The underlying refinement of Catalan numbers is known as the Narayana distribution.

**Proof.** The proof is an immediate combination of Proposition 4 (with  $p = 2$ ) and Lemma 6:

$$\begin{aligned} S_{2,0}(t, v) &= [x^2] \log B(x; t, v) \\ &= [x^2] \log \frac{1}{\sqrt{1 - 4t^2(v + x^2)(v + \bar{x}^2)}} \\ &= \frac{1}{2} [x^2] \sum_{n \geq 1} \frac{4^n t^{2n}}{n} (v + x^2)^n (v + \bar{x}^2)^n \\ &= \frac{1}{2} \sum_{n \geq 1} \frac{4^n t^{2n}}{n} \sum_{m=1}^n v^{2m-1} \binom{n}{m} \binom{n}{m-1}. \end{aligned}$$

This gives the second result of Proposition 7. Now when  $v = 1$ ,

$$B(x; t, 1) = \frac{1}{\sqrt{1 - 4t^2\bar{x}^2(1 + x^2)^2}}$$

and a similar calculation yields the first result of the proposition. ■

## 4 On the steps of Greg Lawler

We consider walks on the slit plane with steps in a generic set  $\mathfrak{S}$ . We denote by  $\overline{\mathfrak{S}}$  the set of steps obtained by reversing the steps of  $\mathfrak{S}$ . That is,  $\overline{\mathfrak{S}} = \{(-i, -j) : (i, j) \in \mathfrak{S}\}$ . Accordingly, the generating functions for walks on the slit plane with steps in  $\mathfrak{S}$  are denoted  $S(x, y; t)$ ,  $\bar{S}_0(x; t)$ , and so on. Observe that the generating functions for loops with steps in  $\mathfrak{S}$  and  $\overline{\mathfrak{S}}$  are equal.

The following proposition relates a number of “difficult” series (counting walks on the slit plane) to the generating function  $B(x; t)$  of bilateral walks, which is simpler to determine (see Lemma 3). We shall see that it contains and extends the cycle lemma result. Note that a *bilateral* walk on the slit plane is simply a walk of  $\mathcal{S}_0$ .

**Proposition 8** *The generating functions for bilateral walks on the slit plane with steps in  $\mathfrak{S}$  (resp.  $\overline{\mathfrak{S}}$ ) and the generating function  $L(t)$  for loops are related by*

$$\bar{S}_0(\bar{x}; t)L(t)S_0(x; t) = B(x; t), \quad (4)$$

where  $B(x; t)$  counts bilateral walks. The generating function for walks on the slit plane (with steps in  $\mathfrak{S}$ ) can be expressed in terms of the generating function for bilateral walks on the slit plane:

$$S(x, y; t) = \frac{S_0(x; t)}{K(x, y; t)B(x; t)},$$

where  $K(x, y; t)$  is the polynomial given by (3).

**Proof.** Let  $w$  be an ordinary walk with steps in  $\mathfrak{S}$ . Let  $(i, 0)$  be the leftmost point of the half-line  $\mathcal{H}$  visited by  $w$ . By looking at the first and last visit of  $w$  to this point, we factor  $w$  into three walks  $w_1, w_2, w_3$ , where  $w_1$  is obtained by reversing a bilateral walk on the slit plane with steps in  $\overline{\mathfrak{S}}$ ,  $w_2$  is a loop and  $w_3$  belongs to  $\mathcal{S}$  (Figure 5). This gives

$$\frac{1}{K(x, y; t)} = \bar{S}_0(\bar{x}; t)L(t)S(x, y; t). \quad (5)$$

By restricting the argument to *bilateral* walks, we obtain the first identity of the proposition. Finally, eliminating  $\bar{S}_0(\bar{x}; t)$  between (4) and (5) gives the second identity of Proposition 8. ■

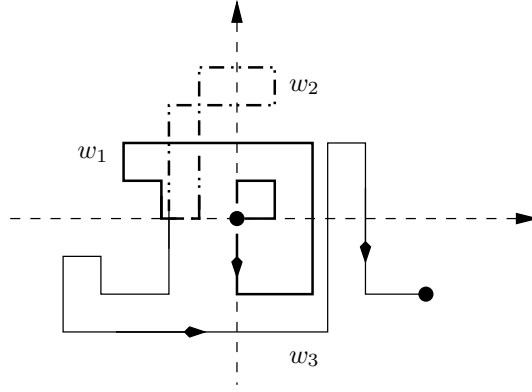


Figure 5: Factorisation of an ordinary walk.

Let us now explain why we claim that the first part of Proposition 8 generalizes the cycle lemma result of Proposition 4. Let us take logarithms in (4):

$$\log \bar{S}_0(\bar{x}; t) + \log L(t) + \log S_0(x; t) = \log B(x; t). \quad (6)$$

The only walk of  $\mathcal{S}_0$  ending at abscissa 0 is the empty walk. Hence

$$S_0(x; t) = 1 + \sum_{i \geq 1} S_{i,0}(t)x^i.$$

For  $i \geq 1$ , let us extract the coefficient of  $x^i$  from (6). The series  $\log \bar{S}_0(\bar{x}; t)$  and  $\log L(t)$  do not contribute, and we obtain the following proposition.

**Proposition 9** *For  $i \geq 1$ , the generating function  $S_{i,0}(t)$  for walks on the slit plane ending at  $(i, 0)$  is  $D$ -finite and can be computed by induction on  $i$  via the following identity:*

$$\sum_{k=1}^i \frac{(-1)^{k-1}}{k} \sum_{\substack{i_1 + \dots + i_k = i \\ i_1 > 0, \dots, i_k > 0}} S_{i_1,0} \cdots S_{i_k,0} = [x^i] \log B(x; t)$$

where  $B(x; t)$  counts bilateral walks.

Let  $p$  be the smallest positive integer such that there exists a walk on the slit plane ending at the point  $(p, 0)$ . The case  $i = p$  of the above proposition is exactly Proposition 4. The argument that proves the holonomy of  $S_{i,0}(t)$  for general  $i$  is the same as in Proposition 4.

Let us illustrate the above result by the example of the ordinary square lattice. For this model,  $p = 1$ , and the first three instances of Proposition 9 are:

$$\begin{aligned} S_{1,0} &= [x] \log B(x; t), \\ S_{2,0} - \frac{1}{2} S_{1,0}^2 &= [x^2] \log B(x; t), \\ S_{3,0} - S_{1,0} S_{2,0} + \frac{1}{3} S_{1,0}^3 &= [x^3] \log B(x; t). \end{aligned}$$

The generating function  $B(x; t)$  for bilateral walks on the square lattice is derived from Lemma 1,

$$B(x; t) = \frac{1}{\sqrt{(1 - t\bar{x}(1+x)^2)(1 - t\bar{x}(1-x)^2)}},$$

and allows us to compute explicitly

$$[x^i] \log B(x; t) = \sum_{n \geq 0} \frac{t^{2n+i}}{2n+i} \binom{4n+2i}{2n}.$$

This series can actually be seen to be algebraic, so that  $S_{i,0}(t)$  is not simply D-finite, but also algebraic. We shall exhibit in Section 7 a whole family of sets  $\mathfrak{S}$  such that all natural families of walks on the slit plane with steps in  $\mathfrak{S}$  have an algebraic generating function.

As we did for the cycle lemma result, we can state explicitly the result on one-dimensional walks that is underlying Propositions 8 and 9. We go back to the framework of Lemma 5.

**Proposition 10** *For  $i \geq 1$ , let  $S_i \equiv S_i(\lambda)$  denote the generating function for walks that start from 0, end at  $i$ , and always have a positive level once they have left their starting point. This series can be computed by induction on  $i$  via the following identity:*

$$\sum_{k=1}^i \frac{(-1)^{k-1}}{k} \sum_{\substack{i_1 + \dots + i_k = i \\ i_1 > 0, \dots, i_k > 0}} S_{i_1} \dots S_{i_k} = [x^i] \log \frac{1}{1 - \sum_{i \in \mathfrak{P}} \lambda_i x^i}.$$

We still have one thing to do in this section: explain its title. The way I discovered the first identity of Proposition 8 is not the one that is presented above. The short and nice proof presented above was “observed” by Gilles Schaeffer, probably less than two minutes after I showed him the identity... The original proof was longer, and was inspired by a probabilistic argument due to Greg Lawler [9], which we describe here in combinatorial terms.

Let  $w$  be a non-empty walk on the slit plane, on the ordinary square lattice. Let  $j = \min\{i \geq 0 : w \text{ visits } (i, 0)\}$ . If  $j = 0$ , then  $w$  completely avoids the  $x$ -axis. Otherwise, we factor  $w$  at the last time it visits  $(j, 0)$  (see Figure 6.a). This gives

$$\mathbb{S} = \mathbb{A} + \mathbb{F} \mathbb{S}, \tag{7}$$

where  $\mathbb{A}$  is the set of walks that never return to the  $x$ -axis and  $\mathbb{F}$  is the set of words  $w$  satisfying:

$$|w|_n = |w|_s, \quad \delta(w) > 0, \quad \text{and for all factorizations } w = uv \text{ such that } u \neq \epsilon, \text{ either } |u|_n \neq |u|_s \text{ or } \delta(v) \leq 0.$$

Let us now consider the set  $\mathcal{W}$  of all square lattice walks. A walk  $w \in \mathcal{W}$  is of one of the following types:

- either  $w$  avoids the half-line  $\mathcal{H}' = \{(k, 0) : k < 0\}$ ; by looking at the last time when  $w$  visits  $(0, 0)$ , we observe that the generating function of such walks is  $\mathbb{L} \mathbb{S}$ , where  $\mathbb{L}$  counts loops,

- or  $w$  meets the half-line  $\mathcal{H}'$ ; in this case, we factor  $w$  at the first time it meets  $\mathcal{H}'$  (Figure 6.b).

This gives

$$\mathbb{W} = \mathbb{L} \mathbb{S} + \bar{\mathbb{F}} \mathbb{W} \quad (8)$$

where  $\bar{\mathcal{F}}$  is the language of words  $w$  satisfying

$$|w|_n = |w|_s, \delta(w) < 0, \text{ and for all factorizations } w = uv \text{ such that } v \neq \epsilon, \text{ either } |u|_n \neq |u|_s \text{ or } \delta(u) \geq 0.$$

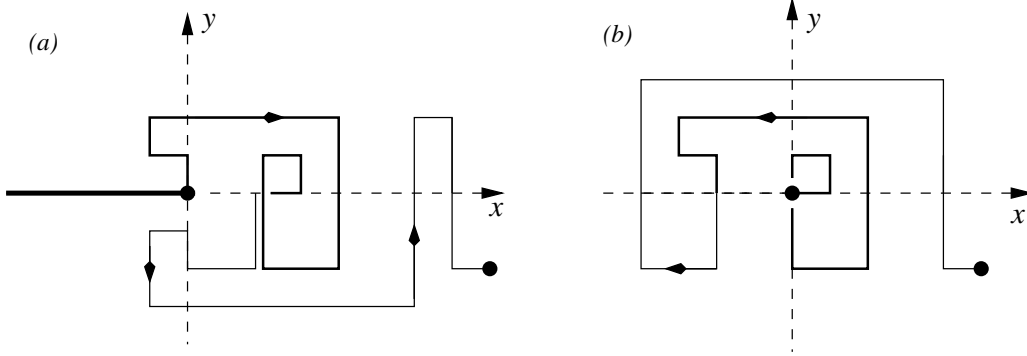


Figure 6: Factorization of words of  $\mathcal{S}$  and  $\mathcal{W}$ .

It is obvious on Figure 6 that the walks of  $\bar{\mathcal{F}}$  are obtained by reversing the direction in walks of  $\mathcal{F}$ . In terms of words, the words of  $\bar{\mathcal{F}}$  are obtained by reading the words of  $\mathcal{F}$  from right to left, replacing each occurrence of  $o$  (resp.  $n, e, s$ ) by a letter  $e$  (resp.  $s, o, n$ ). Naturally, this can also be checked on the definitions of the sets  $\mathcal{F}$  and  $\bar{\mathcal{F}}$ . Let us convert (7) and (8) into identities on length generating functions, then eliminate  $F$ . We obtain:

$$L(t)S(1, 1; t)^2 = A(1, 1; t)W(1, 1; t).$$

But the set  $\mathbb{A}$  of walks avoiding the  $x$ -axis is simply related to the set  $\mathbb{B}$  of bilateral walks: indeed, by looking at the last time an ordinary walk visits the  $x$ -axis, we obtain  $\mathbb{W} = \mathbb{B} \mathbb{A}$ . Hence, by Lemma 1, the above identity reads  $L(t)S(1, 1; t)^2 = (1 - 4t)^{-3/2}$ . This identity, formulated in probabilistic terms, is Proposition 2.4.6 in [9]. It is obviously not sufficient to characterize  $S(1, 1; t)$ , but it is sufficient to prove that the number of  $n$ -step walks on the slit plane grows like  $4^n n^{-1/4}$ , and that's what Lawler does.

Our first variation on Lawler's approach is to take into account the endpoint of the walks; more variables should give more information. Using (7) and (8), we thus obtain:

$$S(\bar{x}, \bar{y}; t)L(t)S(x, y; t) = A(\bar{x}, \bar{y}; t)W(x, y; t).$$

This identity follows from Proposition 8, given that  $W = AB = 1/K$ . Our second variation comes from the observation that it is difficult to work with series that, like  $S(x, y; t)$ , contain both positive and negative powers of  $x$  (and  $y$ ). That is why we focus on the set  $\mathcal{S}_0$  of bilateral walks on the slit plane: their final abscissa is always positive, and their final ordinate is of course 0. The restriction of Lawler's argument to bilateral walks is straightforward. It boils down to imposing this condition to all walks under consideration, thus replacing  $\mathcal{S}$  by  $\mathcal{S}_0$ ,  $\mathcal{A}$  by  $\{\epsilon\}$ , and  $\mathcal{W}$  by  $\mathcal{B}$ , the set of bilateral walks. This is how we obtained the first identity of Proposition 8. Our third and last observation is that this can be done with any set of steps.

## 5 The complete solution

The factorization of  $B(x; t)$  given in Proposition 8, obtained by a combinatorial argument, justifies our interest in the following factorization lemma. This lemma also played a key role in [2], for reasons that were far from being as clear as here.

**Lemma 11 (The factorization lemma [2])** *Let  $B(x; t)$  be a series in  $t$  with coefficients in  $\mathbb{R}[x, \bar{x}]$ , and assume  $B(x; 0) = 1$ . There exists a unique triple  $(B_0(t), B_+(x; t), B_-(\bar{x}; t)) \equiv (B_0, B_+(x), B_-(\bar{x}))$  of formal power series in  $t$  satisfying the following conditions:*

- $B(x) = B_0 B_+(x) B_-(\bar{x})$ ,
- the coefficients of  $B_0$  belong to  $\mathbb{R}$ ,
- the coefficients of  $B_+(x)$  belong to  $\mathbb{R}[x]$ ,
- the coefficients of  $B_-(\bar{x})$  belong to  $\mathbb{R}[\bar{x}]$ ,
- $B_0(0) = B_+(x; 0) = B_-(\bar{x}; 0) = B_+(0; t) = B_-(0; t) = 1$ .

**Proof.** Let us take logarithms in the factorization of  $B$ :

$$\log B_0(t) + \log B_+(x; t) + \log B_-(\bar{x}; t) = \log B(x; t).$$

The conditions  $B_+(0; t) = B_+(x; 0) = 1$  force the series  $\log B_+(x; t)$  to be a multiple of  $x$ . Similarly,  $\log B_-(\bar{x}; t)$  must be a multiple of  $\bar{x}$ . But  $\log B(x; t)$  can be written in a unique way as

$$\log B(x; t) = L_0(t) + xL_1(x; t) + \bar{x}L_2(\bar{x}; t) \quad (9)$$

where  $L_0$  does not depend on  $x$ ,  $L_1$  is a series in  $t$  with polynomial coefficients in  $x$ , and  $L_2$  is a series in  $t$  with polynomial coefficients in  $\bar{x}$ . This forces

$$\log B_0(t) = L_0(t), \quad \log B_+(x; t) = xL_1(x; t) \quad \text{and} \quad \log B_-(\bar{x}; t) = \bar{x}L_2(\bar{x}; t). \quad (10)$$

This shows the existence and uniqueness of the factorization. ■

This lemma implies that the first equation of Proposition 8 characterizes completely the series  $S_0(x; t)$  and  $L(t)$ , and reduces the solution of any slit plane model to the explicit factorization of an algebraic series.

**Theorem 12 (The complete solution)** *Let  $\mathfrak{S}$  be an arbitrary set of steps. Let  $B(x; t)$  be the generating function for bilateral walks with steps in  $\mathfrak{S}$ . The generating function  $S(x, y; t)$ ,  $S_0(x; t)$  and  $L(t)$ , which count various families of walks on the slit plane (general walks, bilateral walks and loops, respectively) can be expressed in terms of the polynomial  $K(x, y; t)$  (given by (3)) and the canonical factorization of  $B(x; t)$ :*

$$\begin{aligned} L(t) &= B_0(t), \\ S_0(x; t) &= B_+(x; t), \\ S(x, y; t) &= \frac{1}{K(x, y; t) B_0(t) B_-(\bar{x}; t)}. \end{aligned}$$

*The logarithm of each of these series is D-finite.*

The holonomy results follows from the first point of Proposition 13 below. More specifically, we shall be concerned in the sequel of the paper with the *algebraic* nature of the three series given by this theorem. We shall exhibit some sets of steps  $\mathfrak{S}$  such that all of them are algebraic (Section 7), and some other sets of steps such that none of them are algebraic, nor even D-finite (Section 8). Hence we need to clarify the nature of the canonical factors of an algebraic series  $B(x; t)$ . The following proposition gives a partial answer.

**Proposition 13 (The nature of the canonical factors)** *Let  $B(x; t)$  be an algebraic series in  $t$  with coefficients in  $\mathbb{R}[x, \bar{x}]$ , such that  $B(x; 0) = 1$ . Let  $(B_0(t), B_+(x; t), B_-(\bar{x}; t))$  be its canonical factorization.*

1. The series  $\log B_0(t)$ ,  $\log B_+(x; t)$  and  $\log B_-(\bar{x}; t)$  are D-finite in all their variables.

2. The series  $B_0(t)$  is D-finite if and only if the series

$$[x^0] \frac{B'(x; t)}{B(x; t)}$$

is algebraic (the derivative is taken with respect to  $t$ ).

3. The series  $B_+(x; t)$  is D-finite in  $t$  if and only if the series

$$[x^>] \frac{B'(x; t)}{B(x; t)} := \sum_{i \geq 1} x^i [x^i] \frac{B'(x; t)}{B(x; t)}$$

is algebraic. In this case, both  $B_-(\bar{x}; t)$  and  $B_0(t)$  are also D-finite in  $t$ .

4. If  $B(x; t)$  is a polynomial of  $\mathbb{R}[x, \bar{x}, t]$ , then  $B_0(t)$ ,  $B_+(x; t)$  and  $B_-(\bar{x}; t)$  are algebraic. Moreover,  $B_+(x; t)$  is a polynomial in  $x$ , and  $B_-(\bar{x}; t)$  is a polynomial in  $\bar{x}$  (with coefficients in  $\mathbb{R}[[t]]$ ).

The proof of this proposition requires the following result.

**Proposition 14 (Singer [16])** *Let  $Y(t)$  be power series in  $t$  with coefficients in a field  $\mathbb{K}$  of characteristic zero. Assume  $Y'(t)/Y(t)$  is holonomic. Then  $Y(t)$  is holonomic if and only if  $Y'(t)/Y(t)$  is actually algebraic.*

It is easy to prove that the algebraicity of  $Y'(t)/Y(t)$  implies the holonomy of  $Y(t)$ , but the converse is significantly more difficult.

### Proof of Proposition 13

1. As in the proofs of Lemma 3 and Proposition 4, we take a new indeterminate  $\bar{x}$ , independent of  $x$ , and form the generating function  $\bar{B}(x, \bar{x}; t)$  of bilateral walks obtained by giving a weight  $x^i t$  to any step  $(i, j)$  such that  $i \geq 0$  and a weight  $\bar{x}^i t$  to any step  $(-i, j)$  such that  $i < 0$ . This series is the diagonal of a rational series and is algebraic over  $\mathbb{Q}(x, \bar{x}, t)$ . As the logarithm of an algebraic series is D-finite,  $\log \bar{B}(x, \bar{x}; t)$  is D-finite in its three variables. By Eqs. (9–10), the series  $\log B_0(t)$  is the diagonal of  $\log \bar{B}(x, \bar{x}; t)$  in  $x$  and  $\bar{x}$ , evaluated at  $x = 1$ . Similarly,  $\log B_+(x; t)$  is the diagonal of  $u/(1-u) \times \log \bar{B}(x, u\bar{x}; t)$  in  $x$  and  $u$ , evaluated at  $\bar{x} = 1/x$ . Finally,  $\log B_-(\bar{x}; t)$  is the diagonal of  $u/(1-u) \times \log \bar{B}(ux, \bar{x}; t)$  in  $\bar{x}$  and  $u$ , evaluated at  $\bar{x} = 1/x$ . The properties recalled in Section 2.3 imply that these three series are D-finite in all their variables.

2. By differentiating the first identity of (10) with respect to  $t$ , we obtain:

$$\frac{d}{dt} \log B_0(t) = \frac{B'_0(t)}{B_0(t)} = \frac{d}{dt} [x^0] \log B(x; t) = [x^0] \frac{\partial}{\partial t} \log B(x; t) = [x^0] \frac{B'(x; t)}{B(x; t)}.$$

This series is clearly D-finite (it is the derivative of a D-finite series). The announced result follows from Proposition 14, applied to  $Y = B_0$  and  $\mathbb{K} = \mathbb{C}$ .

3. Similarly,

$$\frac{B'_+(x; t)}{B_+(x; t)} = [x^>] \frac{B'(x; t)}{B(x; t)},$$

and Proposition 14, applied to  $Y = B_+(x; t)$  and  $\mathbb{K} = \mathbb{C}(x)$ , proves the first statement of point 3. Now with obvious notations,

$$\frac{B'(x; t)}{B(x; t)} = [x^{\leq}] \frac{B'(x; t)}{B(x; t)} + [x^>] \frac{B'(x; t)}{B(x; t)},$$

and  $B'/B$  is algebraic. Hence, if  $[x^>]B'(x; t)/B(x; t)$  is algebraic, then so is  $[x^{\leq}]B'(x; t)/B(x; t)$ . This series is a power series in  $t$  with coefficients in  $\mathbb{R}[\bar{x}]$ . Setting  $\bar{x} = 0$  in this series yields  $[x^0]B'(x; t)/B(x; t)$ , which is, consequently, also algebraic. By difference,  $[x^<]B'(x; t)/B(x; t)$  is also algebraic; finally Proposition 14, used in the easy direction, implies that both  $B_0(t)$  and  $B_-(\bar{x}; t)$  are D-finite in  $t$ .

4. This point was already proved in [2]. We repeat the proof, since we shall use it to solve explicit examples. Let the smallest exponent of  $x$  occurring in  $B(x; t)$  be  $-m$ . Then  $P(x; t) = x^m B(x; t)$  is a polynomial in  $x$  and  $t$  such that  $P(x; 0) = x^m$ . As a polynomial in  $x$ ,  $P$  has degree, say,  $d$ , and hence admits  $d$  roots,

denoted  $X_1, \dots, X_d$ , which are Puiseux series in  $t$  with complex coefficients; in particular, there exists an integer  $n \geq 1$  such that all these roots can be written as Laurent series in the variable  $z = t^{1/n}$  (see [19, Theorem 6.1.5]). Assume that exactly  $k$  of these roots, say  $X_1, \dots, X_k$ , are finite at  $z = 0$ . The other  $d - k$  roots contain terms of the form  $z^{-i}$ , with  $i > 0$ . The polynomial  $P(x; t)$  can be factored as

$$P(x; t) = x^m B(x; t) = \tilde{B}_0(z) \prod_{i=1}^k (x - X_i) \prod_{i=k+1}^d \left(1 - \frac{x}{X_i}\right),$$

where  $\tilde{B}_0(z)$  is, as the  $X_i$ , an algebraic function of  $z$ . For  $i > k$ , the series  $1/X_i$  equals 0 at  $z = 0$ . Hence the condition  $P(x; 0) = x^m$  implies that  $k = m$ , that  $\tilde{B}_0(0) = 1$ , and that the finite roots  $X_1, \dots, X_m$  equal 0 when  $z = 0$ . Let

$$\tilde{B}_+(x; z) = \prod_{i=m+1}^d \left(1 - \frac{x}{X_i}\right) \quad \text{and} \quad \tilde{B}_-(\bar{x}; z) = \prod_{i=1}^m (1 - \bar{x} X_i).$$

Then the series  $\tilde{B}_0(z)$ ,  $\tilde{B}_+(x; z)$  and  $\tilde{B}_-(\bar{x}; z)$  clearly satisfy all the conditions defining the canonical factorization of  $B(x; t)$ , but two: we still need to prove that they are actually series in  $t$  (and not only in  $z = t^{1/n}$ ) with real (rather than complex) coefficients. As in the proof of Lemma 11, we argue that the series  $\log B(x; z^n) = \log B(x; t)$  has a unique decomposition as a sum of three series (one with positive powers of  $z$ , one with negative powers of  $z$ , one with terms independent of  $z$  — see Eq. (9)) to prove that the series  $\tilde{B}_0(z)$ ,  $\tilde{B}_+(x; z)$  and  $\tilde{B}_-(\bar{x}; z)$  are indeed the canonical factors of  $B(x; t)$ . ■

#### Remark: effectiveness of the solution

One might wonder how effective the solution of the slit plane models given by Theorem 12 is. We claim that for any given set  $\mathfrak{S}$ , one can compute algorithmically a linear differential equation (in  $t$ ) satisfied by each of the series  $\log L(t)$ ,  $\log S_0(x; t)$  and  $\log S(x, y; t)$ . The reason for this is that all the results about rational, algebraic and D-finite series that we have used to conclude that  $\log S(x, y; t)$  and its siblings are D-finite are effective, and actually implemented in the MAPLE packages GFUN and MGFUN.

Let us give a few details. One starts from the polynomial  $K(x, y; t)$  given by (3). By converting  $1/K$  into partial fractions of  $y$ , we obtain an expression of  $B(x; t)$  in terms of the solutions  $Y_i$  of  $K(x, Y_i; t) = 0$ . By elimination of the  $Y_i$ , we then obtain a polynomial equation satisfied by the generating function  $B(x; t)$  for bilateral walks. We can just as well compute a polynomial equation for the three-variable series  $\overline{B}(x, \bar{x}; t)$  used in the proof of Proposition 13. That is the first step. An algorithm that derives from the polynomial equation satisfied by  $\overline{B}$  a differential equation (in any of the variables  $x, \bar{x}$  or  $t$ ) satisfied by  $\log \overline{B}$  is implemented in GFUN. The extraction of diagonals is implemented in MGFUN. One thus obtains differential equations for the series  $\log B_0$ ,  $\log B_+(x; t)$  and  $\log B_-(\bar{x}; t)$ , and hence for  $\log L(t)$  and  $\log S_0(x; t)$ . Finally, we need an effective version of the closure of D-finite series under the sum to obtain a differential equation (in  $t$ , say) satisfied by  $\log S(x, y; t)$ : this is provided by the package GFUN.

This holds for any set  $\mathfrak{S}$ . Moreover, we shall study in Section 7 a family of sets  $\mathfrak{S}$  such that the generating function of bilateral walks is of the form  $B(x; t) = P(x; t)^{-1/2}$ . The last point of Proposition 13 tells us how to obtain *explicit expressions* of the canonical factors of  $B(x; t)$  in terms of the roots  $X_i$  of  $P$ .

## 6 A step-by-step approach

Theorem 12 relates three generating functions for walks on the slit plane to the canonical factorization of the series  $B(x; t)$  that counts bilateral walks. In this section, we present another derivation of this theorem. The starting point of this alternative proof is a functional equation satisfied by the generating function  $S(x, y; t)$  that counts all walks on the slit plane. This equation simply translates the fact that a walk on the slit plane is obtained by... adding a step to another walk on the slit plane. This extremely elementary argument, and the corresponding equation, were also the starting point of [2]. But in that paper, we were only able to solve the functional equation when the set  $\mathfrak{S}$  of steps satisfies two simple conditions. Here, we shall solve



the equation by a new and simpler method, which works for any set of steps  $\mathfrak{S}$ . This approach is actually so simple that I find it somewhat mystifying... It seems that we solve an a priori non-trivial problem without any clever combinatorial or algebraic argument. In more positive words, it shows that the crucial ingredient of the solution is the factorization lemma.

**Lemma 15** *Let  $S(x, y; t)$  and  $L(t)$  denote the generating functions for walks on the slit plane and for loops, respectively. There exists a series  $\Omega(\bar{x}; t)$  in  $\mathbb{N}[\bar{x}][[t]]$  such that  $\Omega(\bar{x}; 0) = 0$  and*

$$K(x, y; t)S(x, y; t) = 1 - \Omega(\bar{x}; t), \quad L(t) = \frac{1}{1 - \Omega(0; t)}$$

with, as above,

$$K(x, y; t) = 1 - t \sum_{(i,j) \in \mathfrak{S}} x^i y^j. \quad (11)$$

**Proof.** We obtain an equation for the series  $S(x, y; t)$  by saying that a walk of length  $n$  is obtained by adding a step to another walk of length  $n - 1$ . However, this procedure sometimes produces a *bridge*, that is, a nonempty walk that starts at  $(0, 0)$ , ends on the half-line  $\mathcal{H}$ , but otherwise avoids  $\mathcal{H}$ . Hence, denoting by  $\Omega(\bar{x}; t)$  the generating function for bridges, we have:

$$S(x, y; t) = 1 + tS(x, y; t) \left( \sum_{(i,j) \in \mathfrak{S}} x^i y^j \right) - \Omega(\bar{x}; t),$$

which is exactly the announced equation for  $S(x, y; t)$ . Now a loop is a sequence of bridges, all ending at  $(0, 0)$ . The equation for  $L(t)$  follows. ■

**Corollary 16** *Theorem 12 holds.*

**Proof.** We have established

$$S(x, y; t) = \frac{1 - \Omega(\bar{x}; t)}{K(x, y; t)}. \quad (12)$$

Note that  $1/K(x, y; t)$  counts all walks with steps in  $\mathfrak{S}$ . Extracting the coefficient of  $y^0$  in this equation gives:

$$S_0(x; t) = B(x; t)(1 - \Omega(\bar{x}; t)),$$

where  $B(x; t)$  is the generating function for bilateral walks. We claim that this immediately implies Theorem 12! Indeed, the uniqueness of the canonical factorization of  $B(x; t)$  (Lemma 11), combined with the properties of  $S_0(x; t)$  and  $\Omega(\bar{x}; t)$ , forces

$$S_0(x; t) = B_+(x; t)$$

(one of the three results of Theorem 12), and

$$1 - \Omega(\bar{x}; t) = \frac{1}{B_0(t)B_-(\bar{x}; t)}. \quad (13)$$

The latter identity, plugged in (12), yields

$$S(x, y; t) = \frac{1}{K(x, y; t)B_0(t)B_-(\bar{x}; t)}$$

which is the third statement of Theorem 12. Finally, (13) and Lemma 15 imply that  $L(t) = B_0(t)$ , which completes this new proof of Theorem 12. ■

**Remark: a connection with Spitzer's book**

It is well-known that *everything* is in Spitzer's book, *Principles of random walks* [17]. The results of Sections 5 and 6 are, to a certain extent, not an exception to this rule. In Chapter 4 of his book, Spitzer studies generic one-dimensional random walks  $0 = S_0, S_1, S_2 \dots$  where  $S_i \in \mathbb{Z}$  for all  $i$ . He is interested in the following stopping times:

$$T = \min\{1 \leq n \leq \infty : S_n > 0\} \quad \text{and} \quad T' = \min\{1 \leq n \leq \infty : S_n \geq 0\}.$$

Our two-dimensional combinatorial problem can be turned into a probabilistic one as follows: assume each step of  $\mathfrak{S}$  is taken with probability  $1/k$ , where  $k$  is the cardinality of  $\mathfrak{S}$ . Then, if we only look at the positions where this two-dimensional random walk hits the  $x$ -axis, we effectively obtain a one-dimensional random walk with steps in  $\mathbb{Z}$ . The series  $f_e(t, x)$ ,  $f_i(t, x)$  and  $c(t)$  defined in Definition 17.D3 of [17] (and also used in [8]) are then the counterparts of  $1/B_+(x; t)$ ,  $1/B_-(\bar{x}; t)$  and  $1/B_0(t)$ , respectively. Proposition 17.P4 in [17] is the counterpart of  $B = B_-B_+B_0$ . The first two results of Proposition 17.P5 are related to the enumeration of bridges; more precisely, Eq. 17.P5(b) is the counterpart of Eq. (13) above. The next two results of Proposition 17.8 are related to the enumeration of walks avoiding  $\mathcal{H}$ ; more precisely, Eq. 17.P5(d) is the counterpart of the second identity of Theorem 12. And so on! Of course, there are some differences between our treatment and Spitzer's: we deal with exact enumeration rather than probabilities, and a unit time in our model does not coincide with a unit time in Spitzer's problem. But these differences are minor, and one has to realize that we have (as always?) solved our two-dimensional model by reducing it to a one-dimensional question. What happens between two visits to the  $x$ -axis has no importance. However, what we definitely add to Spitzer's treatment is that our results are effective; the next two sections will provide many explicit examples.

## 7 Walks with small height variation: algebraicity

In this section, we prove that if the set of steps  $\mathfrak{S}$  satisfies the following *small height variation* condition:

$$\text{for all } (i, j) \in \mathfrak{S}, \quad |j| \leq 1,$$

then all the generating functions for walks on the slit plane defined in Section 2.1 are algebraic. This condition prevents a walk from crossing the half-line  $\mathcal{H}$  without hitting it. Three examples will be solved explicitly: the ordinary square lattice and the diagonal square lattice studied in Section 3, and the triangular lattice of Figure 7.

Let  $\mathfrak{S}$  be a set of steps satisfying the small height variation condition. We define three Laurent polynomials in  $x$ , denoted  $A_{-1}(x)$ ,  $A_0(x)$  and  $A_1(x)$ , by

$$A_j(x) = \sum_{(i, j) \in \mathfrak{S}} x^i.$$

The polynomial  $K(x, y; t)$ , given by (11), can be rewritten

$$K(x, y) = 1 - t\bar{y}A_{-1}(x) - tA_0(x) - tyA_1(x).$$

### 7.1 General results

The following result was proved in [2] under the symmetry hypothesis  $A_1(x) = A_{-1}(x)$ . The new approaches developed in this paper show that this assumption is unnecessary.

**Theorem 17** *Let  $\mathfrak{S}$  be a set of steps with small height variations. Let  $\Delta(x; t)$  be the following polynomial in  $x, \bar{x}$  and  $t$ :*

$$\Delta(x; t) = (1 - tA_0(x))^2 - 4t^2A_1(x)A_{-1}(x). \quad (14)$$

*Then the generating function for bilateral walks is*

$$B(x; t) = \frac{1}{\sqrt{\Delta(x; t)}}.$$

Let  $(\Delta_0(t), \Delta_+(x; t), \bar{\Delta}_-(\bar{x}; t))$  be the canonical factorisation of  $\Delta(x; t)$ , defined in Lemma 11. Then the generating function for walks on the slit plane with steps in  $\mathfrak{S}$  is

$$S(x, y; t) = \frac{\sqrt{\Delta_0(t)\Delta_-(\bar{x}; t)}}{K(x, y; t)}.$$

The generating function for bilateral walks on the slit plane is

$$S_0(x; t) = \frac{1}{\sqrt{\Delta_+(x; t)}}. \quad (15)$$

The generating function for loops is

$$L(t) = \frac{1}{\sqrt{\Delta_0(t)}}.$$

All these series are algebraic.

**Proof.** We proceed as in Lemma 6 to enumerate bilateral walks. The generating function  $M(x; t)$  for bilateral walks that stay on or above the  $x$ -axis satisfies

$$\begin{aligned} M(x; t) &= 1 + tA_0(x)M(x; t) + tA_1(x)M(x; t)tA_{-1}(x)M(x; t) \\ &= 1 + tA_0(x)M(x; t) + t^2A_1(x)A_{-1}(x)M(x; t)^2. \end{aligned}$$

The generating function  $B$  for bilateral walks is related to  $M$  by

$$B(x; t) = 1 + tA_0(x)B(x; t) + 2t^2A_1(x)A_{-1}(x)M(x; t)B(x; t).$$

(The factor 2 comes from the fact that primitive bilateral words starting with a down step are obtained by reading from right to left primitive bilateral words starting with an up step.) The expression of  $B(x; t)$  follows. Theorem 12 provides the values of  $S(x, y; t)$ ,  $S_0(x; t)$  and  $L(t)$ . By Proposition 13, the series  $\Delta_0(t)$ ,  $\Delta_+(x; t)$  and  $\Delta_-(\bar{x}; t)$  are algebraic, and so are  $S(x, y; t)$ ,  $S_0(x; t)$  and  $L(t)$ . ■

As in [2], we derive from this central result that the generating function for walks on the slit plane ending at a prescribed position  $(i, j)$  is also algebraic. This, we believe, is a surprising result, as it is not true for ordinary walks on the square lattice, nor for walks avoiding the whole  $x$ -axis.

**Theorem 18** *For all  $j \in \mathbb{Z}$ , the generating function  $S_j(x; t)$  for walks on the slit plane ending at ordinate  $j$  is algebraic. More precisely, for  $j \geq 0$ ,*

$$\begin{aligned} S_j(x; t) &= \frac{1}{\sqrt{\Delta_+(x; t)}} \left( \frac{Z(x; t)}{A_{-1}(x)} \right)^j, \\ S_{-j}(x; t) &= \frac{1}{\sqrt{\Delta_+(x; t)}} \left( \frac{Z(x; t)}{A_1(x)} \right)^j, \end{aligned}$$

where

$$Z(x; t) = \frac{1 - tA_0(x) - \sqrt{\Delta(x; t)}}{2t}.$$

and  $\Delta(x; t)$  is given by (14). The generating function  $S_{i,j}(t)$  for walks on the slit plane ending at  $(i, j)$  is also algebraic for any  $(i, j)$ :

- If  $j = 0$ , Eq. (15) shows that  $S_{i,j}(t)$  belongs to the extension of  $\mathbb{Q}(t)$  generated by the coefficients of  $\Delta_+(x; t)$  (seen as a polynomial in  $x$ ).
- For  $j > 0$ , it belongs to the extension of  $\mathbb{Q}(t)$  generated by  $\sqrt{\Delta_0(t)}$ , the coefficients of  $\Delta_+(x; t)$  and  $\Delta_-(x; t)$  (seen as polynomials in  $x$ ), and also the series  $\sqrt{\Delta_+(\alpha_i; t)}$ , where the numbers  $\alpha_i$  are the roots of the polynomial  $A_{-1}(x)$ , and the algebraic numbers  $\alpha_i$  themselves.

– A similar statement holds if  $j < 0$ , upon replacing  $A_{-1}(x)$  by  $A_1(x)$ .

**Proof.** The proof is very similar to what we did in [2]. The series  $S_j(x; t)$  is the coefficient of  $y^j$  in  $S(x, y; t)$ . To extract this coefficient from the expression of  $S(x, y; t)$  given in Theorem 17, we need to convert the rational function  $1/K(x, y; t)$  into partial fractions of  $y$ . As a polynomial in  $y$ ,  $K(x, y; t)$  has two roots. One of them is  $Z(x; t)/A_1(x)$ , and the other is  $A_{-1}(x)/Z(x; t)$ . Hence

$$\begin{aligned} \frac{1}{K(x, y; t)} &= \frac{yZ(x; t)}{t(yA_1(x) - Z(x; t))(A_{-1}(x) - yZ(x; t))} \\ &= \frac{1}{\sqrt{\Delta(x; t)}} \left( \frac{1}{1 - y\frac{Z(x; t)}{A_1(x)}} + \frac{1}{1 - y\frac{Z(x; t)}{A_{-1}(x)}} - 1 \right). \end{aligned}$$

As  $Z(x; t)$  is a series in  $t$  (with coefficients in  $\mathbb{R}[x, \bar{x}]$ ) such that  $Z(x; 0) = 0$ , each term in the above expression is a well-defined series in  $t$ , and it is now easy to extract the coefficient of  $y^j$  from  $1/K(x, y; t)$ , and hence from  $S(x, y; t)$ . The announced expressions of  $S_j$  and  $S_{-j}$  follow. From this point, the derivation of  $S_{i,j}(t)$  exactly copies what was done in [2, Section 3.2], and we do not repeat the argument. ■

## 7.2 Three examples

The first two examples studied below (the ordinary square lattice and the diagonal square lattice) were already solved in [2].

**Theorem 19 (The ordinary square lattice)** *The generating function  $S(x, y; t)$  for walks on the slit plane, the generating function  $S_0(x; t)$  for bilateral walks on the slit plane and the generating function  $L(t)$  for loops are given by:*

$$\begin{aligned} S(x, y; t) &= \frac{(1 - 2t(1 + \bar{x}) + \sqrt{1 - 4t})^{1/2} (1 + 2t(1 - \bar{x}) + \sqrt{1 + 4t})^{1/2}}{2(1 - t(x + \bar{x} + y + \bar{y}))}, \\ S_0(x; t) &= \frac{2t}{(2t - x(1 - 2t - \sqrt{1 - 4t}))^{1/2} (2t - x(1 + 2t - \sqrt{1 + 4t}))^{1/2}}, \\ L(t) &= \frac{(\sqrt{1 + 4t} - 1)(1 - \sqrt{1 - 4t})}{4t^2} = \sum_{n \geq 0} (2 \cdot 4^n C_n - C_{2n+1}) t^{2n}, \end{aligned}$$

where  $C_n = \binom{2n}{n}/(n+1)$  is the  $n$ th Catalan number. These series are algebraic of degree 8, 8 and 4 respectively.

For any  $(i, j)$ , the generating function  $S_{i,j}(t)$  for walks on the slit plane ending at  $(i, j)$  belongs to  $\mathbb{Q}(t, \sqrt{1 - 4t}, \sqrt{1 + 4t})$ .

**Proof.** We apply Theorem 17 with  $A_0(x) = x + \bar{x}$  and  $A_1(x) = A_{-1}(x) = 1$ . Hence  $\Delta(x; t) = (1 - t(x + \bar{x} + 2))(1 - t(x + \bar{x} - 2))$ . In order to compute the canonical factorization of  $\Delta$ , we apply the procedure described in the proof of Proposition 13. The polynomial  $\Delta$  has four roots  $X_i$ ,  $1 \leq i \leq 4$ , which are quadratic functions of  $t$  and can be computed explicitly. Let  $C(t)$  denote the generating function for Catalan numbers:

$$C(t) = \frac{1 - \sqrt{1 - 4t}}{2t} = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} t^n. \quad (16)$$

Then  $X_1 = C(t) - 1$  and  $X_2 = 1 - C(-t)$  are the two roots that are finite at  $t = 0$ , and by symmetry of  $\Delta$  in  $x$  and  $\bar{x}$ , the two other roots are

$$X_3 = (C(t) - 1)^{-1} = \frac{1 - 2t + \sqrt{1 - 4t}}{2t} \quad \text{and} \quad X_4 = (1 - C(-t))^{-1} = \frac{1 + 2t + \sqrt{1 + 4t}}{2t}. \quad (17)$$

Hence the canonical factorization of  $\Delta$  is such that

$$\Delta_+(x; t) = \Delta_-(x; t) = (1 - xX_3^{-1})(1 - xX_4^{-1}) \quad (18)$$

$$= (1 - x(C(t) - 1))(1 + x(C(-t) - 1)). \quad (19)$$

Taking the coefficient of  $x^2$  in the relation  $\Delta(x) = \Delta_0(t)\Delta_+(x; t)\Delta_-(\bar{x}; t)$  yields

$$\Delta_0(t) = t^2 X_3 X_4 \quad (20)$$

$$= (C(t)C(-t))^{-2}, \quad (21)$$

as  $(C(t) - 1) = tC(t)^2$ . Now by Theorem 17, Eqs. (18) and (20),

$$S(x, y; t) = \frac{\sqrt{t^2 X_3 X_4 (1 - \bar{x} X_3^{-1})(1 - \bar{x} X_4^{-1})}}{K(x, y; t)} = \frac{\sqrt{(tX_3 - t\bar{x})(tX_4 - t\bar{x})}}{1 - t(x + \bar{x} + y + \bar{y})}.$$

The announced expression of  $S$  now follows from (17). Finally, Theorem 17, combined with the explicit expressions (19) and (21), gives the expressions of  $S_0(x; t)$  and  $L(t)$ .

We now apply Theorem 18: the polynomial  $A_1(x) = 1$  having no root, the series  $S_{i,j}(t)$  belongs, by (19) and (21), to  $\mathbb{Q}(t, C(t), C(-t))$ , that is, to  $\mathbb{Q}(t, \sqrt{1 - 4t}, \sqrt{1 + 4t})$ . ■

**Theorem 20 (The diagonal square lattice)** *The generating function  $S(x, y; t)$  for walks on the slit plane, the generating function  $S_0(x; t)$  for bilateral walks on the slit plane and the generating function  $L(t)$  for loops are given by*

$$S(x, y; t) = \frac{(1 - 8t^2(1 + \bar{x}^2) + \sqrt{1 - 16t^2})^{1/2}}{\sqrt{2}(1 - t(x + \bar{x})(y + \bar{y}))}.$$

$$S_0(x; t) = \frac{2\sqrt{2}t}{(8t^2 - x^2(1 - 8t^2 - \sqrt{1 - 16t^2}))^{1/2}},$$

and

$$L(t) = \frac{1 - \sqrt{1 - 16t^2}}{8t^2} = \sum_{n \geq 0} 4^n C_n t^{2n},$$

where  $C_n = \binom{2n}{n}/(n+1)$  is the  $n$ th Catalan number. These series are algebraic of degree 4, 4 and 2 respectively.

For any  $(i, j)$ , the generating function  $S_{i,j}(t)$  for walks on the slit plane ending at  $(i, j)$  belongs to  $\mathbb{Q}(t, \sqrt{1 - 4t}, \sqrt{1 + 4t})$ .

**Proof.** For the diagonal square lattice,  $A_0(x) = 0$  and  $A_1(x) = A_{-1}(x) = x + \bar{x}$ . Again, the polynomial  $\Delta(x) = 1 - 4t^2(x + \bar{x})^2$  has four roots  $X_i$ ,  $1 \leq i \leq 4$ , which are quadratic functions of  $t$  and can be expressed explicitly in terms of the Catalan generating function (16). More precisely,  $X_{1,2} = \pm 2tC(4t^2)$  and

$$X_{3,4} = \pm X_1^{-1} = \pm \frac{1 + \sqrt{1 - 16t^2}}{4t}. \quad (22)$$

We then follow the same steps as above. The formal expression of  $\Delta_+(x; t)$  in terms of the roots  $X_i$  remains unchanged, but its actual value is of course different:

$$\Delta_+(x; t) = \Delta_-(x; t) = (1 - xX_3^{-1})(1 - xX_4^{-1}) \quad (23)$$

$$= 1 - 4t^2 x^2 C(4t^2)^2. \quad (24)$$

Taking the coefficient of  $x^2$  in the relation  $\Delta(x; t) = \Delta_0(t)\Delta_+(x; t)\Delta_-(\bar{x}; t)$  yields now

$$\Delta_0(t) = -4t^2 X_3 X_4 \quad (25)$$

$$= C(4t^2)^{-2}. \quad (26)$$

The expression of  $S(x, y; t)$  now follows from Theorem 17, Eqs. (23), (25) and (22). Finally, Theorem 17, combined with the explicit expressions (24) and (26), gives the expressions of  $S_0(x; t)$  and  $L(t)$ .

We now apply Theorem 18. The extension of  $\mathbb{Q}(t)$  generated by  $\sqrt{\Delta_0(t)}$  and the coefficients of  $\Delta_+(x; t)$  and  $\Delta_-(\bar{x}; t)$  is, by (26) and (24), the field  $\mathbb{Q}(t, C(4t^2)) = \mathbb{Q}(t, \sqrt{1-16t^2})$ . But the polynomial  $A_1(x) = A_{-1}(x) = x + \bar{x}$  has two roots at  $x = \pm i$ , and

$$\sqrt{\Delta_+(\pm i; t)} = \sqrt{1 + 4t^2 C(4t^2)^2} = \sqrt{C(4t^2)} = \frac{\sqrt{1+4t} - \sqrt{1-4t}}{4t},$$

so that the series  $S_{i,j}(t)$  finally belongs to  $\mathbb{Q}(t, \sqrt{1-4t}, \sqrt{1+4t})$ . ■

We conclude this section by a more complicated example: the triangular lattice of Figure 7. Each edge of the lattice can be traversed in both directions, so that the set of steps is  $\mathfrak{S} = \{(0, 1), (1, 0), (1, 1), (0, -1), (-1, 0), (-1, -1)\}$ . Hence  $A_0(x) = x + \bar{x}$ ,  $A_1(x) = 1 + x$  and  $A_{-1}(x) = 1 + \bar{x}$ , so that

$$\Delta(x; t) = (1 - t(x + \bar{x}))^2 - 4t^2(1 + x)(1 + \bar{x}).$$

This polynomial has again four roots, but it is now irreducible, whereas it split into two quadratic terms in the previous examples. We shall express our results in terms of the unique series  $J(t) = t + O(t^2)$  satisfying

$$J = t \frac{1 - 2J + 6J^2 - 2J^3 + J^4}{(1 - J)^2} = t + 5t^3 + 8t^4 + \dots \quad (27)$$

This series can be expressed with radicals:

$$J = 1 + \frac{1 - 2t + \sqrt{(1+2t)(1-6t)}}{4t} \left( 1 - \sqrt{\frac{1 - \sqrt{(1+2t)(1-6t)}}{2t}} \right). \quad (28)$$

Observe that

$$\frac{1 - 2t - \sqrt{(1+2t)(1-6t)}}{8t^2} = \sum_{n \geq 0} M_n 2^n t^n$$

where  $M_n$  is the  $n$ th *Motzkin* number.

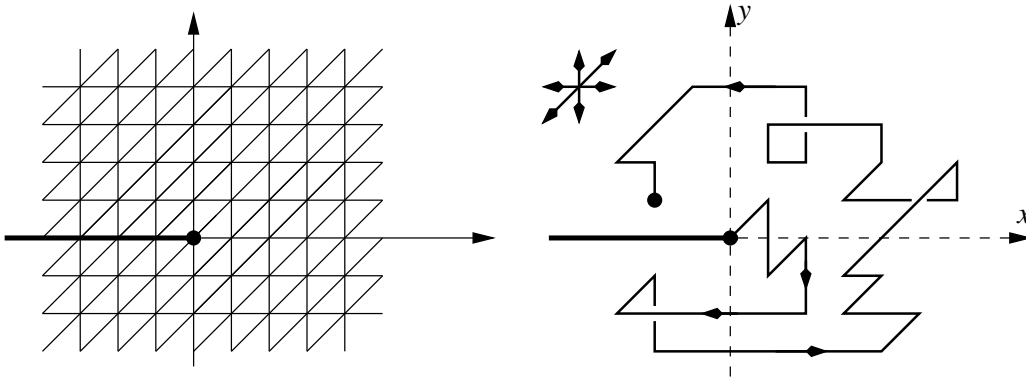


Figure 7: A walk on the slit plane with steps in  $\{(0, 1), (1, 0), (1, 1), (0, -1), (-1, 0), (-1, -1)\}$ .

**Theorem 21 (The triangular lattice)** *The generating function  $S(x, y; t)$  for walks on the slit plane, the generating function  $S_0(x; t)$  for bilateral walks on the slit plane and the generating function  $L(t)$  for loops are given by*

$$S(x, y; t) = \frac{t\sqrt{(1-J)^2 - 2J(1+J^2)\bar{x} + J^2(1-J)^2\bar{x}^2}}{J(1-J)(1-t(x+\bar{x}+y(1+x)+\bar{y}(1+\bar{x})))},$$

$$S_0(x; t) = \frac{1-J}{\sqrt{(1-J)^2 - 2J(1+J^2)\bar{x} + J^2(1-J)^2\bar{x}^2}},$$

and

$$L(t) = \frac{J}{t},$$

where  $J$  is the series in  $t$  defined by (28). These series are algebraic of degree 8, 8 and 4 respectively.

For any  $(i, j)$ , the generating function  $S_{i,j}(t)$  for walks on the slit plane ending at  $(i, j)$  belongs to  $\mathbb{Q}(J)$ .

**Proof.** As  $\Delta(x; t)$  is quadratic in  $x + \bar{x}$ , its four roots can be easily expressed in terms of square roots. One finds:

$$X_1 = \frac{1+2t+2\sqrt{t(1+3t)} - \sqrt{(1+2t)(1+6t+4\sqrt{t(1+3t)})}}{2t} = t - 2t\sqrt{t} + 2t^2 + \dots,$$

$$X_2 = \frac{1+2t-2\sqrt{t(1+3t)} - \sqrt{(1+2t)(1+6t-4\sqrt{t(1+3t)})}}{2t} = t + 2t\sqrt{t} + 2t^2 + \dots,$$

$$X_3 = \frac{1}{X_1} = \frac{1+2t+2\sqrt{t(1+3t)} + \sqrt{(1+2t)(1+6t+4\sqrt{t(1+3t)})}}{2t} = \frac{1}{t} + \frac{2}{\sqrt{t}} + 2 + \dots,$$

$$X_4 = \frac{1}{X_2} = \frac{1+2t-2\sqrt{t(1+3t)} + \sqrt{(1+2t)(1+6t-4\sqrt{t(1+3t)})}}{2t} = \frac{1}{t} - \frac{2}{\sqrt{t}} + 2 + \dots$$

Hence, once again,

$$\Delta_+(x; t) = \Delta_-(x; t) = (1 - xX_3^{-1})(1 - xX_4^{-1}) = (1 - xX_1)(1 - xX_2). \quad (29)$$

Taking the coefficient of  $x^2$  in the relation  $\Delta(x; t) = \Delta_0(t)\Delta_+(x; t)\Delta_-(\bar{x}; t)$  yields now  $\Delta_0(t) = t^2X_3X_4$ . By Theorem 17, the generating function for loops is thus  $L(t) = \sqrt{X_1X_2}/t$ . Rather than juggling with the expressions of the  $X_i$ 's, we can obtain directly an equation for  $L(t)$  by eliminating  $X_1$  and  $X_2$  in the equations

$$\Delta(X_1; t) = 0, \quad \Delta(X_2; t) = 0, \quad t^2L(t)^2 = X_1X_2.$$

The resulting equation factors into several polynomials. The fact that  $L(t) = 1 + 5t^2 + O(t^3)$  tells us which of these factors cancels  $L(t)$ , and we actually end up with  $L(t) = J/t$ . That is,

$$X_1X_2 = J^2 \quad \text{and} \quad \Delta_0(t) = \frac{t^2}{J^2}. \quad (30)$$

The same elimination method provides

$$X_1 + X_2 = \frac{2J(1+J^2)}{(1-J)^2}.$$

Consequently, the last expression of  $\Delta_+(x; t)$  given in (29) reads

$$\Delta_+(x; t) = 1 - 2x \frac{J(1+J^2)}{(1-J)^2} + x^2J^2, \quad (31)$$

and the expression of  $S_0(x; t)$  follows, using Theorem 17. Finally, by Theorem 17 and (29),

$$S(x, y; t) = \frac{\sqrt{\Delta_0(t)\Delta_+(\bar{x}; t)}}{K(x, y; t)}$$

and the announced expression follows, using (30) and (31).

We now apply Theorem 18. The extension of  $\mathbb{Q}(t)$  generated by  $\sqrt{\Delta_0(t)}$  and the coefficients of  $\Delta_+(x; t)$  and  $\Delta_-(\bar{x}; t)$  is, by (30), (31) and (27), the field  $\mathbb{Q}(J)$ . The polynomials  $A_1(x) = 1 + x$  and  $A_{-1}(x) = 1 + \bar{x}$  have a root at  $x = -1$ , and

$$\sqrt{\Delta_+(-1; t)} = \frac{1 + J^2}{1 - J},$$

so that the series  $S_{i,j}(t)$  belongs to  $\mathbb{Q}(J)$ . ■

## 8 Transcendental examples

### 8.1 Extensions of the $(\pm 1, \pm 1)$ model

We prove in this section that having small *down* steps is not enough to ensure algebraicity. We choose the set of steps  $\mathfrak{S} = \{(1, k), (-1, k), (1, -1), (-1, -1)\}$ , with  $k \geq 1$ . The smallest positive abscissa of bilateral walks is

$$p = \begin{cases} 1 & \text{if } k \text{ is even} \\ 2 & \text{if } k \text{ is odd.} \end{cases}$$

Moreover, a bilateral walk with  $m$  up steps must have  $mk$  down steps, so that its total length is  $n = m(k+1)$ . The case  $k = 1$  was solved explicitly in Theorem 20, and led to algebraic generating functions. This is no longer true when  $k \geq 2$ .

**Proposition 22** *Let  $\mathfrak{S} = \{(1, k), (-1, k), (1, -1), (-1, -1)\}$ , with  $k \geq 2$ . Let  $n$  be a multiple of  $k+1$ , say  $n = m(k+1)$ . If  $k$  is even, assume, moreover, that  $n$  is odd. Then the number of  $n$ -step walks on the slit plane ending at  $(p, 0)$  is*

$$a_{p,0}(n) = \frac{k+1}{m} \binom{n}{\frac{n+p}{2}} \sum_{i=0}^{m-1} \binom{n-1}{i} k^{m-i-1}. \quad (32)$$

*The generating function  $S_{p,0}(t)$  for these walks is  $D$ -finite, but transcendental. The generating functions  $L(t), S_0(1; t)$  and  $S(1, 1; t)$  are not even  $D$ -finite.*

Once again, the proof of this result requires to determine the generating function  $B(x; t)$  for bilateral walks. We notice that  $\mathfrak{S}$  is a product:  $\mathfrak{S} = \{-1, 1\} \times \{-1, k\}$ . This implies that

$$B(x; t) = \tilde{B}((x + \bar{x})t), \quad (33)$$

where  $\tilde{B}(t)$  is the generating function for bilateral walks with steps in  $\{(0, -1), (0, k)\}$ .

**Lemma 23** *Let  $U \equiv U(t)$  be the unique power series in  $t$  satisfying*

$$U = t(1 + U)^{k+1}.$$

*Then the generating function for bilateral walks with steps in  $\{(0, -1), (0, k)\}$  is*

$$\tilde{B}(t) = \frac{1 + U(t^{k+1})}{1 - kU(t^{k+1})}.$$

**Proof.** We begin with the generating function  $\tilde{M}(t)$  for bilateral walks that never go below the  $x$ -axis. Such walks start with a  $(0, k)$  step. By looking at the first time they visit a point at ordinate  $k-1, k-2, \dots, 1, 0$ , we obtain, for their non-commutative generating function,

$$\tilde{M} = \epsilon + n(\tilde{M}s)^k \tilde{M}$$

(the letter  $n$  stands for a north step  $(0, k)$ , and the letter  $s$  for a south step  $(0, -1)$ ). Hence the associated length generating function satisfies

$$\tilde{M}(t) = 1 + t^{k+1} \tilde{M}(t)^{k+1} \quad (34)$$



or, equivalently,

$$\tilde{M}(t) = 1 + U(t^{k+1}). \quad (35)$$

Now, take a bilateral walk  $w$ , and look at the first (positive) time  $t$  where its ordinate is non-negative. Let  $\ell$  denote this ordinate: then  $0 \leq \ell \leq k$  (Figure 8). By looking at the last time before  $t$  when the ordinate of  $w$  is  $0, -1, -2, \dots, \ell - k$ , then at the first time (after  $t$ ) when the ordinate is  $\ell - 1, \dots, 1, 0$ , we obtain:

$$\tilde{\mathbb{B}} = \epsilon + \sum_{\ell=0}^k (s\Phi(\tilde{\mathbb{M}}))^{k-\ell} n(\tilde{\mathbb{M}}s)^\ell \tilde{\mathbb{B}},$$

where  $\Phi(w)$  is the word obtained by reading  $w$  from right to left. Hence

$$\tilde{B}(t) = 1 + (k+1)t^{k+1}\tilde{M}(t)^k\tilde{B}(t),$$

and the announced expression follows from (34) and (35). ■

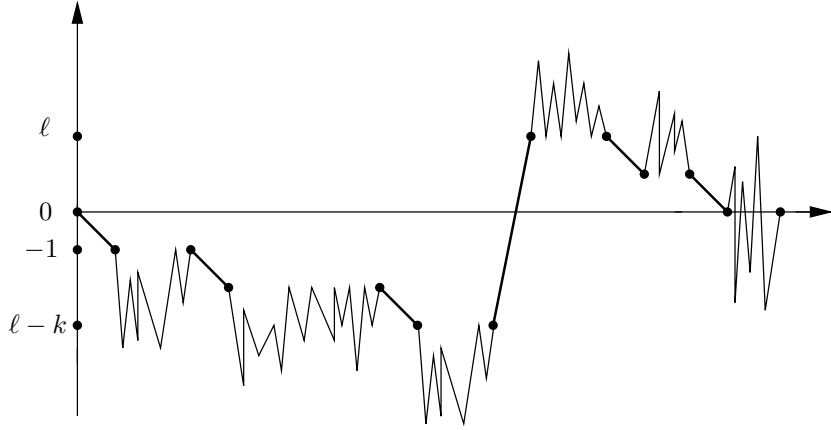


Figure 8: Factorisation of a bilateral walk.

**Proof of Proposition 22.** By Proposition 4, the number of  $n$ -step walks on the slit plane ending at  $(p, 0)$  is

$$a_{p,0}(n) = [x^p t^n] \log B(x; t). \quad (36)$$

By (33) and the above lemma, we are led to compute, using the Lagrange inversion formula,

$$\begin{aligned} [t^m] \log \frac{1+U}{1-kU} &= \frac{k+1}{m} [t^{m-1}] \frac{(1+t)^{m(k+1)-1}}{1-kt} \\ &= \frac{k+1}{m} \sum_{i=0}^{m-1} \binom{(k+1)m-1}{i} k^{m-i-1}. \end{aligned}$$

The announced expression of  $a_{p,0}(n)$  follows.

The series  $S_{p,0}(t)$  is D-finite by Proposition 4. To prove that it is not algebraic, we are going to study the asymptotic behaviour of its coefficients. We shall obtain

$$a_{p,0}(n) = \frac{k+1}{\sqrt{2\pi}} \left( \frac{2(k+1)}{k^{k/(k+1)}} \right)^n n^{-3/2} \left( 1 - \frac{\sqrt{2}(k-1)}{3\sqrt{\pi}k} n^{-1/2} + O(n^{-1}) \right), \quad (37)$$

and the term in  $n^{-2}$  in this expansion makes it incompatible with an algebraic generating function (see [5]). We start from the expression of  $a_{p,0}(n)$  we have just obtained, Eq. (32). An estimate of the (almost) central binomial coefficient can be obtained by standard tools, like Stirling formula:

$$\binom{n}{(n+p)/2} = \sqrt{\frac{2}{\pi}} 2^n n^{-1/2} (1 + O(n^{-1})). \quad (38)$$

The rest of the expression of  $a_{p,0}(n)$  is the coefficient of  $t^m$  in  $\log(1+U)/(1-kU)$ . This series fits perfectly in the framework of algebraic-logarithmic series developed in [7] or [6]. Recall the equation satisfied by  $U$ :

$$P(t, U) := U - t(1+U)^{k+1} = 0. \quad (39)$$

The singularities of  $U$  are to be found at the points  $t$  such that

$$\frac{\partial P}{\partial u}(t, U) = 1 - t(k+1)(1+U)^k = 0.$$

This implies that  $U$  has a unique singularity, at

$$t_c = \frac{k^k}{(k+1)^{k+1}},$$

and that  $U(t_c) = 1/k$ . The series  $U(t)$  admits an analytic continuation in the domain  $D = \mathbb{C} \setminus [t_c, +\infty[$ . The same holds for  $V = (1+U)/(1-kU)$ , because  $U = 1/k$  forces  $t = t_c$ . The logarithm being an analytic function in  $\mathbb{C} \setminus ]-\infty, 0]$ , we have to mind for the values of  $t$  for which  $V(t)$  is real and negative. If  $V(t)$  is real, then so are  $U(t)$ , and  $t$  itself (by (39)). Hence all singularities of  $\log V$  lie on the real line. Now  $V$  never vanishes on the domain  $D$ : this would mean that  $1+U = 0$ , so that, by (39),  $U = 0$ , a contradiction. On  $] -t_c, t_c[$ ,  $V(t)$  is real, given by its series expansion. Moreover,  $V(0) = 1$ , so that by continuity,  $V(t)$  is real positive on all the interval  $] -t_c, t_c[$ . By continuity at  $-t_c$ ,  $V(-t_c)$  is real and non-negative. But  $V$  does not vanish, hence  $V(-t_c) > 0$ . By continuity again, there exists  $t_0 > t_c$  such that for  $t > -t_0$ ,  $V(t) \notin \mathbb{R}^-$ .

We have thus proved that  $\log V$  is not singular in  $\mathbb{C} \setminus (]-\infty, -t_0] \cup [t_c, +\infty[)$ . In particular, we can apply to this series the analysis of singularities of [6]. From the local expansion of  $U$  around its singularity,

$$U(t) = \frac{1}{k} - \frac{\sqrt{2(k+1)}}{k\sqrt{k}} \sqrt{1-t/t_c} + \frac{2(k+2)}{3k^2} (1-t/t_c) + O((1-t/t_c)^{3/2}),$$

we obtain

$$\log V(t) = \log \frac{1+U}{1-kU} = \frac{1}{2} \log \frac{k+1}{2k} + \frac{1}{2} \log \frac{1}{1-t/t_c} + \frac{\sqrt{2}(k-1)}{3\sqrt{k}(k+1)} \sqrt{1-t/t_c} + O(1-t/t_c).$$

The analytic properties of  $\log V$ , derived above, allow us to extract term by term from this estimation the asymptotic behaviour of the coefficient of  $t^m$  in  $\log V$ :

$$[t^m] \log \frac{1+U}{1-kU} = \frac{1}{2m} t_c^{-m} - \frac{k-1}{3\sqrt{2\pi k}(k+1)} t_c^{-m} m^{-3/2} + O(t_c^{-m} m^{-5/2}).$$

Eq. (37) is obtained by combination of this estimate with (38). The transcendence of  $S_{p,0}(t)$  follows.

We now turn our attention to the series  $L(t)$ ,  $S_0(1; t)$  and  $S(1, 1; t)$ . By Theorem 12 and Proposition 13,  $L(t)$  is D-finite if and only if the coefficient of  $x^0$  in  $B'/B$  is algebraic. But

$$[t^{n-1}][x^0] \frac{B'(x; t)}{B(x; t)} = n[t^n][x^0] \log B(x; t),$$

and an asymptotic analysis very similar to the one above (one simply has to replace  $p$  by 0) shows that this coefficient as a term in  $1/n$  in its asymptotic expansion, which prevents the series  $[x^0]B'/B$  from being algebraic. Hence  $L(t)$  is not D-finite.



**Proposition 25** *The length generating function  $S_{1,0}(t)$  for walks on the upper half-plane that start from  $(0,0)$ , end at  $(1,0)$  and avoid the forbidden half-line  $\{(k,0) : k \leq 0\}$  has the following simple expansion:*

$$S_{1,0}(t) = \sum_{n \geq 0} \frac{t^{2n+1}}{2n+1} \binom{2n+1}{n}^2.$$

*It is D-finite, but transcendental. The series  $L(t)$ ,  $S_0(1;t)$  and  $S(1,1;t)$  are not D-finite.*

**Proof.** The counterpart of Proposition 4 is:

$$S_{1,0}(t) = [x] \log M(x;t).$$

This series can be evaluated using the Lagrange inversion formula (LIF). Let  $N(x;t) = tM(x;t)$ . Then

$$N = t(1 + xN)(1 + \bar{x}N),$$

so that  $\log M(x;t) = \log(1 + xN)(1 + \bar{x}N)$ . Using the LIF, we find

$$\log M(x;t) = \sum_{n \geq 1} \frac{t^n}{n} \sum_{k=0}^n \binom{n}{k}^2 x^{2k-n}.$$

Extracting the coefficient of  $x$  yields the announced result for  $S_{1,0}$ . The coefficient of  $t^{2n+1}$  in the series  $S_{1,0}(t)$  grows like  $4^{2n}/n^2$ , up to a multiplicative constant, and this behaviour is not compatible with an algebraic generating function (see [5]). The rest of the argument follows the same principle as the end of the proof of Proposition 22:  $L(t)$  is D-finite if and only if the coefficient of  $x^0$  in  $M'/M$  is algebraic. But

$$[t^{n-1}][x^0] \frac{M'(x;t)}{M(x;t)} = n[t^n][x^0] \log M(x;t) = \binom{n}{n/2}^2 \sim c 4^n n^{-1},$$

and this cannot be the asymptotic behaviour of the coefficients of an algebraic series. The second equation of Proposition 24 shows that  $S(1,1;t)$  is D-finite if and only if  $S_0(1;t)$  is D-finite. As above, this assumption forces  $L(t)$  to be D-finite as well, which is not the case. ■

## 9 Further directions

### 9.1 Towards a complete classification of the sets of steps

In view of the results of Sections 7 and 8, we conjecture that the slit plane model associated with a set of steps  $\mathfrak{S}$  has an algebraic generating function if and only if the  $x$ -axis cannot be crossed without being effectively visited. This general statement does not apply to pathological sets  $\mathfrak{S}$  in which no step goes, for instance, right (or left, or north, or south): the corresponding models are, in essence, one-dimensional, and yield algebraic series. Our conjecture means that Theorem 17 essentially encapsulates all algebraic cases (if we assume the g.c.d. of all vertical moves to be 1). Section 8 suggests that a possible approach of this conjecture lies in the asymptotic study of the coefficients of  $S_{p,0}(t) = [x^p] \log B(x;t)$ . The case where  $\mathfrak{S}$  is a product set  $\mathcal{H} \times \mathcal{V}$  looks promising, since it decouples the problem into one-dimensional problems: in this case,

$$B(x;t) = \tilde{B}(H(x)t)$$

where  $H(x)$  is a polynomial in  $x$  and  $\bar{x}$  describing the horizontal moves and  $\tilde{B}(t)$  is the generating function for one-dimensional bilateral walks with steps in  $\mathcal{V}$ . Hence by Proposition 4,

$$[t^n]S_{p,0} = [x^p]H(x)^n[t^n] \log \tilde{B}(t).$$

As counting one-dimensional bilateral walks consists in evaluating a coefficient of the form  $[y^0]V(y)^n$ , where  $V(y)$  is a Laurent polynomial in  $y$ , this case is likely to be attacked by a classical saddle-point analysis.

## 9.2 Walks on the square lattice avoiding a half-line of rational slope

In the myriad of sets  $\mathfrak{S}$  one could decide to study in greater detail, some are more appealing than the others; for instance, the sets  $\mathfrak{S}_{p,q} = \{(1, -p), (-1, p), (0, q), (0, -q)\}$  which stem from the enumeration of *square lattice* walks avoiding a half-line of rational slope.

Indeed, let  $p$  and  $q$  be two relatively prime integers such that  $p \geq 0$  and  $q > 0$ . We consider walks on the ordinary square lattice that start from the origin, but otherwise never meet the integer points of the half-line  $\mathcal{H}_{p/q} = \{(x, px/q), x \leq 0\}$  (Figure 10). The transformation  $(i, j) \mapsto (i, qj - pi)$  shows that counting these walks is equivalent to counting walks on the slit plane with steps in  $\mathfrak{S}_{p,q}$ . The machinery developed in this paper applies, but several natural questions could be explored further:

- How explicit can the result be made? for slope 2? for a generic slope  $p/q$ ?
- If the slope of the forbidden half-line is 0 or 1, then the generating function one obtains is algebraic, as shown by Theorems 19 and 20. In accordance with the above conjecture, we are tempted to believe that these are the only algebraic cases — but this needs a proof! A possible approach would be to start from Proposition 4: the number of  $n$ -step walks ending at  $(q, p)$  is obtained by extracting a coefficient from  $B(x; t)$ , and one could try to study the asymptotic behaviour of this number.
- It is much easier for a square lattice walk to avoid the integers points of the half-line  $\mathcal{H}_{100}$  than those of  $\mathcal{H}_0$ . I thought for a while that this might affect significantly the asymptotic behaviour of the number of  $n$ -step walks avoiding these lines: this number grows like  $4^n n^{-1/4}$  when  $\mathcal{H}_0$  is forbidden, and I thought that the exponent  $-1/4$  could be replaced by a larger one for  $\mathcal{H}_{100}$ . This intuition was wrong: from the equation

$$S(1, 1, t)^2 L(t) = \frac{1}{(1 - 4t)^2 B(1; t)}$$

derived from Proposition 8, one can prove, as Lawler did for the slope 0 case, that *for any rational slope  $p/q$ , the number of  $n$ -step walks avoiding the integer points of  $\mathcal{H}_{p/q}$  grows like  $4^n n^{-1/4}$*  (see [9, Section 2.4]). This, of course, only holds for rational slopes: if the slope is irrational, the walks only have to avoid the point  $(0, 0)$ : their number is directly related to the return time of a random walk and grows like  $4^n / \log(n)$  (see [17, p. 167]).

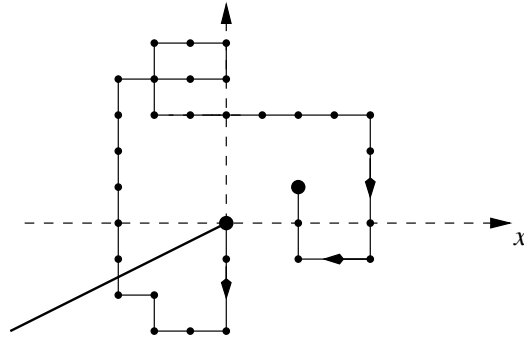


Figure 10: This path crosses the half-line of slope  $r = 1/2$ , but has no vertex on it.

## 9.3 Real avoidance of the half-line

When we extended the slit plane model from square lattice walks to walks with steps in an arbitrary set  $\mathfrak{S}$ , we only forbade the walks to have a vertex on the half-line  $\mathcal{H}$ . The reason for this choice was simply that the methods we had developed for the original case worked perfectly for this extension of the model. But what if we also forbid the walks to *cross* the half-line? For sets of steps with small height variations, the model is unchanged, and yields an algebraic generating function. What about other sets of steps? Can one

of the methods presented here (or both) be adapted to this new convention? What can be said of the nature of the associated generating functions?

## 9.4 Higher dimensions

Let  $\mathfrak{S}$  be a finite subset of  $\mathbb{Z}^d$ , with  $d \geq 2$ . We consider  $d$ -dimensional walks that start from the origin of the lattice  $\mathbb{Z}^d$  and have their steps in  $\mathfrak{S}$ . The generating function for these walks, counted by their length (variable  $t$ ) and position of their endpoint (variables  $x_1, \dots, x_d$ ) is

$$W(x_1, \dots, x_d; t) = \frac{1}{K(x_1, \dots, x_d; t)}$$

with

$$K(x_1, \dots, x_d; t) = 1 - t \sum_{(i_1, \dots, i_d) \in \mathfrak{S}} x_1^{i_1} \cdots x_d^{i_d}.$$

A walk is said to be *bilateral* if it ends on the  $x_1$ -axis, that is, on the (one-dimensional) line  $\{(k, 0, \dots, 0), k \in \mathbb{Z}\}$ . The generating function for bilateral walks is

$$B(x_1; t) = [x_2^0 \cdots x_d^0] W(x_1, \dots, x_d; t).$$

This series is the iterated diagonal of a rational function: hence it is always D-finite, but usually transcendental when  $d \geq 3$ . We consider walks that never return to the half-line  $\mathcal{H} = \{(k, 0, \dots, 0), k \leq 0\}$  once they have left their starting point. All the machinery developed in Sections 4 and 5 remains valid. In particular, Proposition 8 and Theorem 12 still hold, so that the enumeration of these walks boils down to the explicit factorization of a *D-finite* (rather than algebraic) series, namely  $B(x_1; t)$ . How explicitly can this be done?

Let us examine the simplest case: walks on the cubic lattice  $\mathbb{Z}^3$ , with steps in  $\{(0, 0, 1), (0, 1, 0), (1, 0, 0), (0, 0, -1), (0, -1, 0), (-1, 0, 0)\}$ . The projection of a bilateral walk on the  $yz$ -plane is a two-dimensional walk that starts and ends at the origin. The number of such walks with  $2n$  steps is easily seen to be  $\binom{2n}{n}^2$ . Hence the generating function for bilateral walks on the cubic lattice is

$$B(x; t) = \sum_{N \geq 0} t^N \sum_{n=0}^{N/2} \binom{N}{2n} \binom{2n}{n}^2 (x + \bar{x})^{N-2n}.$$

Using the algorithm described in [13, Chap. 6], and implemented by Zeilberger in the MAPLE package EKHAD, one can derive from the above expression a linear recurrence relation satisfied by the coefficients of  $t^n$  in  $B(x; t)$ ; this recurrence relation can then be translated into a following differential equation of order 2. What can be said of the canonical factors of  $B(x; t)$ ? Can they also be described by differential equations? Note that the number of  $n$ -step walks avoiding  $\mathcal{H}$  is known to grow like  $6^n / \sqrt{\log n}$  (see Eq. (2.35) in [9]).

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